Section 30 Properties of the Riemann Integral.

Theorem 30.1 A monotone function on [a,b] is Riemann integrable.

Proof (Case of monotone increasing)

We apply Theorem 29.9 Riemann Integrability Critereon.

The function *f* is bounded (Why?)

Given $\varepsilon > 0$, we have $\dfrac{\varepsilon}{f(b) - f(a)} > 0$ (Why)

We can thus choose *k* such that $k < \dfrac{\varepsilon}{f(b) - f(a)}$ (Why)

Thus $0 < k(f(b) - f(a)) < \varepsilon$.

Let P = $\{x_0, x_1, \ldots, x_n\}$ be a partition of [*a*,*b*] that has each subinterval of length $(x_i - x_{i-1}) \leq k$.

Then since *f* is increasing, $m_i = f(x_{i-1})$ and $M_i = f(x_i)$.

Then $U(f,P) - L(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i (x_i - x_{i-1})$

$$= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))k$$

$$= k\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$

$$= k(f(b) - f(a)) < \varepsilon$$

Thus the integrability criterion is satisfied, and the monotone increasing function is integrable.