Section 29 The Riemann Integral.

Definitions:

- **Partition** of a closed interval: P is a partition of [a,b] if P = {x_0, x_1, …, x_n} where $a = x_0 < x_1 < \ldots < x_n = b$.

- **Refinement** of a partition:

A partition Q if a **refinement** of the partition P if P is a subset of Q.

- **Upper and Lower Sums:**

Let f be a bounded function on [a,b] and P = {x_0, x_1, …, x_n **}** a partition of [a.b]

- **Upper Darboux Sum of *f* with respect to P is defined as**

$$U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]$

- **Lower Darboux Sum of *f* with respect to P is defined as**

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]$

- **Upper Integral** U(f) = inf{U(f,P): P is a partition of [a,b])

- **Lower Integral** L(f) = sup{L(f,P): P is a partition of [a,b])

Riemann Integrable The function f is Reimann integrable on [a,b] if L(f) = U(f) and this common value is then called the Reimann integral of f on [a,b] and is denoted by

$$\int_a^b f(x)dx$$

Geometric interpretation (Figure 29.1): Area of approximating rectangles.

Theorem 29.4: If *f* is a bounded function and P and Q are partitions of [a,b] and Q is a refinement of P, then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

This is geometrically justified in figure 29.2 (page 271)

Theorem 29.6: $L(f) \le U(f)$

Proof (using idea of practice 29.5)

For *any* two partitions P and Q of [a,b], the union of these two $P \cup Q$ is a refinement of each.

Hence:

$$L(f,P) \le L(f,P \cup Q) \le U(f,P \cup Q) \le U(f,Q)$$

or

$$L(f,P) \le U(f,Q)$$

This makes U(*f*,Q) an upper bound on {L(*f*,*P*): P is a partition of [*a*,*b*])}.

Thus

$$L(f) = sup\{L(f,P): \text{P is a partition of } [a,b])\} \le U(f,Q) \text{ for any partition Q.}$$

This then means that

$$L(f) \le U(f) = inf\{U(f,Q): \text{Q is a partition of } [a,b])\}$$

Theorem 29.9 Riemann Integrability Critereon. A function *f* defined and bounded on [a,b] is Riemann integrable if and only if for every there exists a partition P of [*a*,*b*] such that U(*f*,P) – L(*f*,P) < ε.

Proof:

Part I --
Assume that *f* is Riemann integrable.

Then L(f) = U(f)

Since L(*f*) = *sup*{L(*f*,P): P is a partition of [*a*,*b*]},
we can find a partition P_1 such that L(*f*,P_1) > L(*f*) – ε/2 (Why?)

Since U(*f*) = *inf*{U(*f*,P): P is a partition of [*a*,*b*]},
we can find a partition P_2 such that U(*f*,P_2) < U(*f*) + ε/2 (Why?)

Form the partition $P = P_1 \cup P_2$.

Then

$$U(f,P) - L(f,P) \le U(f,P_2) - L(f,P_1) < [U(f) + \frac{\varepsilon}{2}] - [L(f) - \frac{\varepsilon}{2}] = U(f) - L(f) = \varepsilon$$

Part II: (Converse).

Assume that given any ε >0 there is a partition P of [a,b] such that U(*f*,P) – L(*f*,P) < ε, or

U(*f*,P) < L(*f*,P) + ε,

Then

$$U(f) \le U(f,P) \le L(f,P) + \varepsilon \le L(f) + \varepsilon$$

Since this means $U(f) \le L(f) + \varepsilon$ holds for every ε> 0, we have

$$U(f) \le L(f)$$

We already know by Thoerem 29.6 that $L(f) \le U(f)$, so they are in fact equal

By definition, the function is Riemann integrable.