Section 25 The Derivative

Definition 25.1: Let *f* be a real-valued function defined on an interval I containing the point c. The function *f* is differentiable at *c* if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite.

We can use the sequential criterion for limits (Theorem 20.8, page 193) to get a similar result for derivatives.

Theorem 25.3: Let *f* be a real-valued function defined on an interval I containing the point c. The function *f* is differentiable at *c* if and only if for every sequence (*xn*) in I converging to c (with $x_n \neq c$), the sequence

$$\frac{f(x_n) - f(c)}{x_n - c}$$

converges (and the limit of the sequence will be the derivative $f'(c)$).

This is sometimes useful for proving *nondifferentiability* (i.e. by constructing a sequence x_n that converges to c, but such that $\frac{f(x_n) - f(c)}{x_n - c}$ diverges.

Example 25.4 gives an example of a nondifferentiable function ($f(x) = |x|$).

The sequence $x_n = \frac{(-1)^n}{n}$ converges to 0.

For **even *n*** the value of the sequence $\frac{f(x_n) - f(0)}{x_n - 0} = 1$.

For **odd *n*** the value of sequence $\frac{f(x_n) - f(0)}{x_n - 0} = -1$.

Theorem 25.6: If a function is diff. at a point then it is also continuous at that point.

Proof:

Write

$$f(x) = (x-c)\frac{f(x)-f(c)}{x-c} + f(c)$$

Taking limits:

$$\lim_{x-c} f(x) = \lim_{x-c}(x-c)\lim_{x-c}\frac{f(x)-f(c)}{x-c} + \lim_{x-c} f(c) = f(c)$$

So by the limit test *f* is continuous at *c*.

Theorem 25.7 Properties of Differentiable Functions

Suppose *f* and *g* are real value functions defined on an interval I, and c is a point in I. Then

a) For *k* an real number, $(kf)'(c) = kf'(c)$ b) $(f+g)'(c) = f'(c) + g'(c)$

c) $(fg)'(c) = f(c)g'(c) + g(c)f'(c)$ d) $(\frac{f}{g})'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$

Proof of part c)

We write

$$\frac{(fg)(x)-(fg)(c)}{x-c} = \frac{f(x)g(x)-f(c)g(c)}{x-c}$$

$$= \frac{f(x)g(x)-f(x)g(c)+g(c)f(x)-f(c)g(c)}{x-c}$$

$$= f(x)\frac{g(x)-g(c)}{x-c} + g(c)\frac{f(x)-f(c)}{x-c}$$

Then taking limits:

$$\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c} =$$

$$= \lim_{x\to c} f(x)\lim_{x\to c}\frac{g(x)-g(c)}{x-c} + g(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$= f(c)g'(c) + g(c)f'(c)$$

Example 25.8 uses Mathematical Induction and Theorem 25.7 to show that if f the function defined by $f(x) = x^n$, where n is an integer > 0, then
$f'(x) = nx^{n-1}$

Practice 25.9 extends the result to negative integers of n by using the quotient rule.