Section 21 Continuous Functions

Informal Definition of Continuous Function: Let $f: D \to R$. The function ***f*** at *c* if as points in D get closer and closer to c, the function evaluated at those points gets closer and closer to the functions value at *c*, *f*(*c*).

Formal Definition of Continuisty

The function f is continuous at a point *c* of its domain D, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $x \in D$ and $|x - c| < \delta$.

The definition can be restated in terms of neighborhoods and also in terms of sequences (which follows directly from the work in the previous section on limits).

Theorem 21.2: The function *f* is *continuous* at c if and only if

(*b*) If for every ε – neighborhood V of *f*(c), there exists a δ-neighborhood U of *c* such that $f(U \cap D) \subseteq V$.

(*c*) If for every sequence (x_n) in D that converges to *c*, the sequence $f(x_n)$ converges to *f*(*c*).

(d) If *c* is an accumulation point, continuity at *c* is equivalent to $\lim_{x \to c} f(x) = f(c)$

The negation of (c) can be used to show discontintuity:

Theorem 21.6 The function $f: D \to R$ is discontinuous at c in D if there exists a sequence (x_n) in D that converges to *c* but sequence $f(x_n)$ does not converge to *f*(*c*).

Theorem 21.10: (This theorem extends properties of limits (from Theorem 17.1) to corresponding properties of continuous functions)

Let $f: D \to R$ and $g: D \to R$ both be continuous at c . Then

$f + g$ is continuous at c.

fg is continuous at c.

f/g is continuous at c, if *g*(c) is not 0.

Theorem 21.12: This theorem tells us that the composition of continuous functions is continuous.

Theorem 21.14: The function $f: D \to R$ is continuous if and only if for every open set *G*, there exists an open set H such that H∩D = $f^{-1}(G)$.

Informally this says: A function is continuous iff the preimage of any open set is open.

Recall: A set S is open if for every point c in the set, there exists a neighborhood U of the point c that is completely within S (i.e. a subset of S)

Proof of Theorem:

(Part I: If f is continuous, then for every open set *G*, there exists an open set H such that H∩D = $f^{-1}(G)$. **)**

Let f be continuous on D and let G be an open subset of real numbers.

Let c be the pre-image of a point in G (i.e. $f(c) \in G$)

Since G is open, there is a neighborhood V of f(c) such that $V \subseteq G$.

Using the "neighborhood" version of continuity, there exists a neighborhood U of *c* such that $f(U \cap D) \subseteq V$.

Let H be the union of such neighborhoods over all pre-image points c of points in G.

H is open (since it is the union of open sets).

(Proof of converse)

Let V be a neighborhood of a point c in the domain D.

By the hypotheses of theorem, there is an open set H such that $H \cap D \subseteq f^{-1}(V)$.

Since *f*(*c*) is in V, the point c is in H.

Since H is an open set, there is a neighborhood U of *c* such that U is a subset of H.

Thus $f(U \cap D) \subseteq f(H \cap D) = V$, and *f* is continuous.

See **Example in figure 21.5** for a discontinuous function where preimage of an open set is not open.