Section 20 Limits of Functions

Informal Definition of Limit: Let $f : D \to R$. L is ***the limit of f*** at c if as points in D get closer and closer to c, the function evaluated at those points gets closer and closer to L.

We write this as $\lim_{x \to c} f(x) = L$

Formal Definition of limit:

L is the limit of *f* at an accumulation point *c* of D, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$.

The definition excludes $x = c$ since $0 < |x - c|$ so $f(c)$ can be anything -- it may not even be defined, yet the limit can still exist.

The definition can be restated in terms of neighborhoods:

> L is the limit of *f* at an accumulation point *c* of D, if for every ε – neighborhood V of L, there exists a δ-neighborhood U of *c* such that $f(U \cap D) \subseteq V$

Template for Proving $\lim_{x \to c} f(x) = L$

Given ε > 0, choose δ = _____________. (*This δ will depend on ε.)*

Then for any x in D such that $0 < |x - c| < \delta$ = _______, we have

| f(x) - *L* | = …..

(via algebra ……)

< ε

By definition, $\lim_{x \to c} f(x) = L$

Examples 20.5 and **20.6** provide some standard examples of providing proof of a limit.

Given any $\varepsilon > 0$, we must meet the challenge of a $\delta > 0$ (that will depend on ε).

Theorem 20.8 Let $f : D \to R$ and c be an accumulation point in D. Then $\lim_{x \to c} f(x) = L$ if and only if for every sequence (s_n) in D that converges to *c* (with for all n, the sequence *f*(s_n) converges to L.

This allows us to equivalently define limit of a function in terms of convergent sequences.

Definition 20.12 provides "pointwise definitions" for sum *f* + *g* , product *fg*, quotient *f/g* and scalar multiple *kf* of functions.

> For example, the function f + g is the function whose value at *x* is defined as (*f* + *g*)(*x*) = *f*(*x*) + *g*(*x*)

We can use Theorem 20.8 to obtain the corresponding properties for limits of sums, products, quotients and scalar multiples of functions corresponding to the results for limits of sequences in Theorem 17.1:

Theorem 20.13 (first part) Let $f : D \to R$ and $g : D \to R$, c an accumulation point in D. Then if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, we have $\lim_{x \to c} (f + g)(x) = L + M$

Proof:

Consider any sequence (s_n) in D that converges to *c*.

By Theorem 20.8, *f*(s_n) converges to L and *g*(s_n) converges to M.

By Theorem 17.1,
(*f* + *g*)(s_n) = *f*(s_n)+ *g*(s_n) converges to L + M

Again by Theorem 20.8 (its an if and only if theorem), this means

$$\lim_{x \to c} (f + g)(x) = L + M$$

Class Practice: Construct a proof $\lim_{x \to c} (fg)(x) = LM$

Theorem 20.8 can be used to determine limits of functions as in Practice 20.15 and Example 20.16 – see exercises also.

One-Sided Limits:

L is the right-limit of *f* at an accumulation point *c* of D, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $| f(x) - L | < \varepsilon$ whenever $x \in D$ and $c < x < c + \delta$.

We write this as $\lim_{x \to c+} f(x) = L$