Section 18 Monotone Sequences and Cauchy Sequences

Def.: a) A sequence (s_n) is (monotone) increasing if $s_n \leq s_{n+1}$ for all n.
b) A sequence (s_n) is (monotone) decreasing if $s_n \geq s_{n+1}$ for all n.

18.3 Monotone Convergence Theorem: Every bounded monotone sequence converges.
i.e. If a sequence is monotone and bounded, then it converges.

Note: We already know that if a sequence converges, then it is bounded. (This does not require that the sequence be monotone – Theorem 16.13).

Proof of the Monotone Convergence Theorem

(1) Assume that the sequence is monotone increasing, and bounded.

(2) Let S = { s_n , n = 1, 2, …}

(3) Since S is nonempty and bounded it has a least upper bound, s = sup(S)

(4) Then lim s_n = s, since:

(4a) Given any $\varepsilon > 0$, since $s - \varepsilon$ is not an upper bound for S, there is an element in S, say s_N such that

$$s - \varepsilon < s_N$$

(4b) Since the sequence is increasing

$$s_N < s_n \text{ for all } n > N$$

(4c) Since s is an upper bound for the set of values of the sequence,

$$s_n \leq s \text{ for all } n.$$

(4d) Combining the above three inequalities we have

$$s - \varepsilon < s_N < s_n \leq s \text{ for all } n > N.$$

(4e) Hence we have by definition lim s_n = s.

Exercise: Complete the proof by showing that is the sequence is decreasing, it converges using a greatest lower bound argument.

Theorem 18.8
a) An unbounded increasing sequence diverges to $+\infty$.
b) An unbounded decreasing sequence diverges to $-\infty$.

Proof of a).

(1) Let S = { s_n , n = 1, 2, …}.

(2) S is unbounded above.

(3) Thus given any real M, we can find an N such that $s_N > M$.

(4) Since the sequence is increasing, $s_n > M$ for all $n > N$.

(5) By definition, the sequence diverges to $+\infty$.

In Class Exercise: write out the proof for part b)

Definition: Cauchy Sequence. A sequence is a Cauchy sequence if for every $\varepsilon > 0$ there exists an N such that $| s_n - s_m | < \varepsilon$, whenever $n, m >= N$.

18.10 Lemma: Every convergent sequence is a Cauchy sequence.

Proof:

(1) Suppose that s_n convergs to s.

(2) Given an $\varepsilon > 0$, choose N such that $| s_k – s | < \varepsilon/2$ for all $k > N$

(3) Then $| s_n - s_m | = | s_n – s + s - s_m |$

$$< | s_n – s | + | s - s_m |$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for all n, m} > N$$

(4) By definition the sequence (s_n) is Cauchy.

18.12 THEOREM

(Cauchy Convergence Criterion) A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof:

We have already shown (Lemma 18.10) that a convergent sequence is a Cauchy Sequence.

For the converse we suppose that (s_n) is a Cauchy sequence and let $S = \{s_n : n \in N\}$ be the range of the sequence.

We consider two cases, depending on whether *S* is finite or infinite.

Case 1:

(1) If *S* is finite, then the minimum distance ε between distinct points of *S* is positive.

(2) Since (s_n) is Cauchy, there exists a number *N* such that $m, n > N$ implies that $|s_n - s_m| < \varepsilon.$

(3) Let n_0 be the smallest integer greater than *N*.

(4) Given any $m > N$, s_m and s_{n_0} are both in *S*, so if the distance between them is less than ε, it must be zero (since ε is the minimum distance between *distinct* points in *S*).

(5) Thus $s_m = s_{n_0}$ for all $n > N$. It follows that $\lim s_n = s$.

Case 2: Now suppose that S is infinite.

(6) From Lemma 18.11 we know that S is bounded.

(7) Thus from the Bolzano-Weierstrass Theorem (14.6) there exists a point s in $\Re$ that is an accumulation point of S.

(8) We claim that (s_n) converges to s since:

(9) Given any $\varepsilon > 0$, there exists a number N such that $|s_n - s_m| < \varepsilon/2$ whenever $m, n > N$.

(10) Since s is an accumulation point of S, the neighborhood $N(s;\varepsilon/2) = (s-\varepsilon/2, s+\varepsilon/2)$ contains infinitely many points of S.

(11) Thus in particular there exists an integer $m > N$ such that $s_m \in N(s;\varepsilon/2)$.

(12) Hence for any $n>N$ we have

$$|s_n - s| = |s_n - s_m + s_m - s|$$
$$\leq |s_n - s_m| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(13) Therefore, $\lim s_n = s$.