Lemma 14.4 If S is a nonempty closed, bounded subset of R, then S has a maximum and a minimum.

Proof:

(1) By the Completenss Axiom:
Since S is nonempty and bounded, S has a supremum, m = sup(S).

(2) Suppose that m is not a member of S. We will arrive at a contradiction:

(a) Consider any neighborhood N(m,ε) of m.

(b) There must be an x in S that is greater than m - ε. (Why?)

(c) Also x must be less than m. (Why?)

(d) Thus x is a point in S that is in N*(m, ε).

(e) This means that m is an ____________________.

(f) But since S is ____________, m is in S, a contradiction.

(3) Thus m is a member of S, and S has a max.

The argument for a minimum would be similar.