# Mathematical Induction

- Method of proving that a proposition $P(n)$ is true for *every* natural number $n$.
- Principle of Mathematical Induction

(A rule of inference)

$$\frac{\begin{array}{l} P(1) \\ \forall k \geq 1,\ P(k) \rightarrow P(k+1) \end{array}}{\therefore \forall n \geq 1,\ P(n)}$$

**Problem:** Prove using mathematical induction:

For all natural numbers n, 1 + 2·2 + 3·22 + … + n·2*n* = (n-1)2*n* +*1*

***Brainstorming:***

P(0):

P(*k*):

P(*k*+1):

We can P(*k*+1) from P(*k*) by adding the k+1'st term

to each side of the equality for P(k),

then simplifying the right hand side of this equality.

***Proof:***

Let P(*n*) denote the proposition that

________________________________________

1. Basis Step: P(0) is true, since

________________________________________

2. Inductive Step:

By the P.M.I, ____________________

An inductive proof always requires the following two steps:

**1. The Basis for Induction (Inductive Step):**

Prove P(1) is true

( Usually very easy)

**2. The Induction Step:** Prove P(*k*) $\rightarrow$ P(*k*+1) is always true whenever *k* is a natural number.

Prove the Induction Step by assuming P(*k*) is be true and showing that P(*k* + 1) is true (usually with a direct proof ).

•Note: P(*k*) is called the *induction hypothesis*).

•Note: If you are trying to prove $\forall$ integers n >= 0, P(n), then the Basis of Induction is to prove P(0) instead of P(1).

Example: Prove the following: For all natural numbers n,
$1 + 3 + 5 + \ldots + (2n-1) = n^2$

**Proof:**

Let $P(n)$ denote the proposition that $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

**1. Basis Step:**

$P(1)$ is true since $1 = 1^2$

**2. Inductive Step:**

Assume $P(k)$ is true, that is

(*) $1 + 3 + 5 + \ldots + (2k-1) = k^2$

Adding $2(k+1) - 1$ to each side of (*) gives

$$1 + 3 + 5 + \ldots + (2k-1) + [2(k+1) - 1] = k^2 + [2(k+1) - 1]$$

$$= k^2 + 2k + 1 \quad \text{(simplify)}$$

$$= (k + 1)^2 \quad \text{(factor)}$$

so $P(k+1)$ is true.

By the principle of mathematical induction,
$P(n)$ is true for all integers $n \geq 1$.

Q.E.D.

**Hints for figuring out the inductive step**:
Write down what P(*k*) is. Write down what P(*k* + 1) is.
Try to figure out how they differ -- i.e. how to derive P(*k* + 1) from P(*k*).

**1. P(*k*) is an equality involving a sum of *k* terms**:
The inductive step is usually proven by adding the *k*+1'st term
in the sum to each side of the equality P(*k*).

Then show that gives you P(*k*+1) by simplifying the right hand side.

**2. P(*k*) is an inequality**
You may need to do one or more of the following to show P(*k*+1)
a) Add a number (or inequality in the same direction) to each side.
b) Multiply each side by a positive number, or an inequality.
c) Divide each side by a positive number, or an inequality.