The Heine Borel Theorem

A set is compact if and only if it is closed and bounded.

***Part One**: If S is compact then S is closed and bounded.*

Proof:
(1) Assume S is compact.

Then

a) S is ***bounded*** since:

(2) Consider the intervals $I_n = (-n, n), n = 1,2,...$

(3) Then $R = \bigcup_{n=1}^{\infty} I_n$ is an open covering of S.

(4) Since S is compact, we can select a finite subcovering for S, say
$I_{n_1}, I_{n_2}, \ldots, I_{n_k}$.

(5) Let m be the maximum of the $n_1, \cdots n_k$.
Then S is bounded above by *m* and below by *–m*. (Why?)

b) S is ***closed*** since:
(We show S not closed leads to a contradiction.)

(6) If S is not closed then there is a boundary point *p* of S that is not in S. (Why?)

(7) Let $N_n = [p - \frac{1}{n}, p + \frac{1}{n}]$.

(8) Construct the open covering of S as $U_n = R \setminus N_n$

so $\bigcup_{n=1}^{\infty} U_n = R \setminus \{p\} \supseteq S$

(9) Since S is compact, we select a finite subcover

$$U_{n_1}, U_{n_2}, \ldots, U_{n_k}$$

(10) Let *m* be the maximum of the $n_1, \cdots n_k$.

Then $S \subseteq U_m$

(11) This means that $S \cap (R \setminus U_m) = S \cap [p - \frac{1}{m}, p + \frac{1}{m}] = \phi$, and thus also

$$S \cap (p - \frac{1}{m}, p + \frac{1}{m}) = \phi$$

(12) This contradicts p being a boundary point of S. (Why?)

(13) Thus S must be closed! (Why?)

Part Two (The harder part): If S is closed and bounded, then S is compact.

Proof:

(14) We need to show every open cover of S admits a finite subcover.

(15) Consider any open cover $\Im$ of S. For every real number *x*, construct the set

$$S_x = S \cap (-\infty, x]$$

and let $B = \{x : S_x \text{is covered by a finite subcover of } \Im\}$.

We need to show that
(16) (*) B is nonempty and unbounded above.

This will allow us to complete our argument as follows:

(17) Since B is unbounded, there is a *p* in B that is greater than sup (S) (Why?)

(18) But then S = S_p and S has a finite subcovering.

(19) By definition S is then compact.

So: It remains to show
(*) B is nonempty and unbounded above.

We do this as follows:

(17) Since S is closed and bounded, S has a minimum *d* (why?)

(18) Then $S_d = \{d\}$ and clearly is covered by a finite subcover and thus *d* is in B.

(19) Suppose B were bounded above.

Then $m = \sup B$ exists. (Why?)

(20) Either *m* is a member of S or *m* is not a member of S. We show in either case we get a contradiction of *m* being the supremum of B.

(Case 1) If *m* is a member of S, then there is an open set F_0 from the family $\Im$ that contains *m*.

(21) Since F_0 is open we can find $x_1 < m < x_2$ where $[x_1, x_2]$ is in F_0. (Why)

(22) Since $x_1 < m$, x_1 is a member of B and S_{x1} has a finite covering

$F_1, F_2, \ldots F_k$.

(23) $F_1, F_2, \ldots F_k$ together with F_0 make a finite covering of S_{x2} .

(24) This would make x_2 a member of B, a contradiction of $m = \sup B$.

(Case 2) If *m* is NOT a member of S, since S is closed we can find a neighborhood $N(m,\varepsilon)$ of *m* that is completely outside of S -- i.e. such that $N(m,\varepsilon) \cap S = \Phi$. (Why?)

(25) This would mean $S_{m-\varepsilon} = S_{m+\varepsilon}$. (Why)

(26) Then $m + \varepsilon$ would be in B, a contradiction of *m* being an upper bound for B.

Thus B in fact must be bounded above, and the proof is complete.