Definition: Convergence Sequence: A sequence converges to *s* if for every ε > 0 there exists an N such that $| s_n - s | < \varepsilon$, for all n >= N. (The *s* is called the limit of the sequence.)

Template for Proving Convergence of a Sequence

Given ε > 0, choose N = ____________. (*This N will depend on ε.)*

Then for any *n* > N, we have

$| \quad s_n \quad - \quad s \quad | \; = \quad \ldots..$

(via algebra ……)

$< \varepsilon$

By definition, $\lim \; s_n \; = s$

Example 16.6: Practice for computing N to meet ε challenge.

$$s_n = \frac{n^2 + 2n}{n^3 - 5}$$

Obtain upper bound for numerator and lower bound for denominator

Theorem 16.8 applies method of 16.6 to achieve a more general result:

Let s_n and a_n be sequences, s a real number, and lim $a_n = 0$.

If there exists a $k > 0$ and a natural number m such that

$$|s_n - s| \leq k\,|a_n| \quad \text{for all } n > m$$

then lim $s_n = s$.

Proof:

Given any $\varepsilon > 0$

Since lim $a_n = 0$, there exists an N_1 such that $|a_n| < \varepsilon / k$.

Choose N = max(m, N_1)

Then $|s_n - s| \leq k\,|a_n| < k(\varepsilon / k) = \varepsilon$ for all n > N.

By definition lim $s_n = s$.

Convergence Sequence: A sequence converges to s if for every $\varepsilon > 0$ there exists an N such that $| s_n - s | < \varepsilon$, for all $n > N$.

A sequence **diverges** if no such s exists.

Example 16.12 Show that the sequence defined by $s_n = 1 + (-1)^n$ diverges.

Proof: (Show that s_n converging to s leads to a contradiction.)

Assume that $\lim_{n \to \infty} s_n = s$.

Choose $\varepsilon = 1$ for the challenge in the definition.

Then there must be an N such that $| s_n - s | < 1$ for all $n > N$

For n odd this means $| 0 - s | < 1$ so $-1 < s < 1$

For n even this means $| 2 - s | < 1$ so $1 < s < 3$.

Since both inequalities cannot be satisfied there is no s satisfying $\lim_{n \to \infty} s_n = s$ and the series diverges.

Triangle Inequality for Real Numbers (From Section 11)

$| x + y | < |x| + |y|$

Reverse Triangle Inequality

$|x| - |y| < | x - y |$

Theorem 16.13 Every convergent sequence is bounded.

Proof:

Suppose the sequence $\{ s_n \}$ converges.

Letting $\varepsilon = 1$, we have

$| s_n - s | < 1$ if $n >= N$

Using the Reverse Triangle Inequality, we have:

$| s_n | - | s | < |s_n - s| < 1$ for all $n >= N$

so $| s_n | < | s | + 1$ for all $n >= N$

Define $M = \max\{| s_1 |, | s_2 |, .. | s_{N-1} |, | s | + 1\}$.

Then $| s_n | < M$ for all n, and the sequence is bounded by this M.