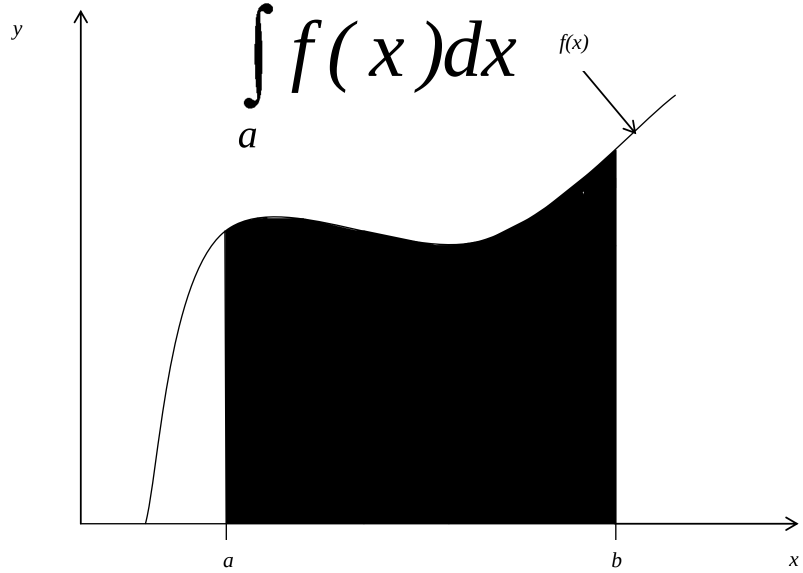
Chapter 7 Quadrature (Numerical Integration)

Integration as Area under Curve

$$\int_a^b f(x)\,dx$$



Quadrature Formulas

Definition 7.1. Suppose that $a = x0 < x1 < \cdots < xM = b$. A numerical integration or quadrature formula is of the form
$Q[f] = w_0 f(x_0) + w_1 f(x_1) + \cdots + w_M f(x_M)$
with the property that

$$\int_a^b f(x) = Q[f] + E[f]$$

The term $E[f]$ is called the
truncation error for integration.
The $\{x_k\}$ are the quadrature nodes and the $\{w_k\}$ are the weights.

Degree of Precision of Quadrature

Definition 7.2. The ***degree of precision*** of a quadrature formula is the positive integer n such that error $E[P_i] = 0$ for all polynomials $P_i(x)$ of degree $i \leq n$, but for which $E[P_{n+1}] = 0$ for some polynomial $P_{n+1}(x)$ of degree $n + 1$.

Newton Cotes Quadrature Formulas

1. Trapezoidal Rule: $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}(f_0 + f_1)$

Precision: n=1 Error Term: $-\frac{h^3}{12} f^{(2)}(c)$

2. Simpson's Rule: $\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2)$

Precision: n = 3 Error Term: $-\frac{h^5}{90} f^{(4)}(c)$

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3. Simpson's 3/8 Rule $$\int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$$

Precision: n =3 Error: $$-\frac{3h^5}{80} f^{(4)}(c)$$

4. Boole's Rule $$\int_{x_0}^{x_4} f(x)dx \approx \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)$$

Precision: n = 5 Error: $$-\frac{8h^7}{945} f^{(6)}(c)$$

Method of Derivation of the Formulas

- Take the appropriate Lagrange Interpolating Polynomial:
- Degree 1 for Trapezoid (two points)
- Degree 2 for Simpson (three points)
- Degree 3 for Simpson 3/8 rule (four points)
- Degree 4 for Boole's Rule (five points)
- Integrate this polynomial. Simplify the integration by using an appropriate change of variables (see text, page 357).

Composite Trapezoidal Rule

Suppose that the interval [a, b] is subdivided into M subintervals [x_k , x_{k+1}] of width $h = (b-a)/M$ by using the equally spaced nodes $x_k = a + kh$, for $k = 0, 1, \ldots, M$. The ***Composite Trapezoidal Rule for M subintervals*** is obtained by applying the Trapezoidal Rule to each of the subintervals:

$$T(f,h) = \frac{h}{2}\sum_{k=1}^{M}(f(x_k) - f(x_{k-1}))$$

This approximation for the integral has error which is $O(h^2)$

$$E_T(f,h) = -\frac{(b-a)f^{(2)}(c)h^2}{12}$$

Composite Simpson's Rule

Suppose that $[a, b]$ is subdivided into $2M$ subintervals $[x_k , x_{k+1}]$ of equal width $h = (b-a)/(2M)$ by using $x_k = a+kh$ for $k = 0, 1, . . . , 2M$.

The ***Composite Simpson rule for 2M subintervals*** is obtained by applying the Simpson's Rule to adjacent each pair of subintervals and is given by

$$S(f,h) = \frac{h}{3}\sum_{k=1}^{M}(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k}))$$

This approximation for the integral has error which is $O(h^4)$:

$$E_S(f,h) = -\frac{(b-a)f^{(4)}(c)h^4}{180}$$