Definition of Taylor Series and Taylor Polynomials

- The Taylor series centered at x_o for a function f that is infinitely differentiable on an interval is given by

$$f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots$$

- The Taylor polynomial of degree n centered at x_o for a function f is given by

$$f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Recursive Computation of Taylor Polynomials.

$$p_0 = f(x_0)$$

$$p_1 = p_0 + f'(x_0)(x - x_0)$$

$$p_2 = p_1 + \frac{f''(x - x_0)}{2!}(x - x_0)^2$$

$$p_3 = p_2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

Theorem: Taylor Polynomial Approximation

Assume that *f* is a function whose first n+1 derivatives exist on an interval (a,b) containing x_o and P_N denotes the nth degree Taylor polynomial centered at x_o.

Then ( ) $\frac{\qquad}{(\qquad)}$, for some c between x and x_o.

( ) $\frac{\qquad}{(\qquad)}$ is called the *remainder (or error) term*.

Example of Application of Taylor Approximation

Consider the Taylor Series for e^x centered at 0.

— —+…

If we approximate *e* by evaluated at 1 with 15 terms we get:

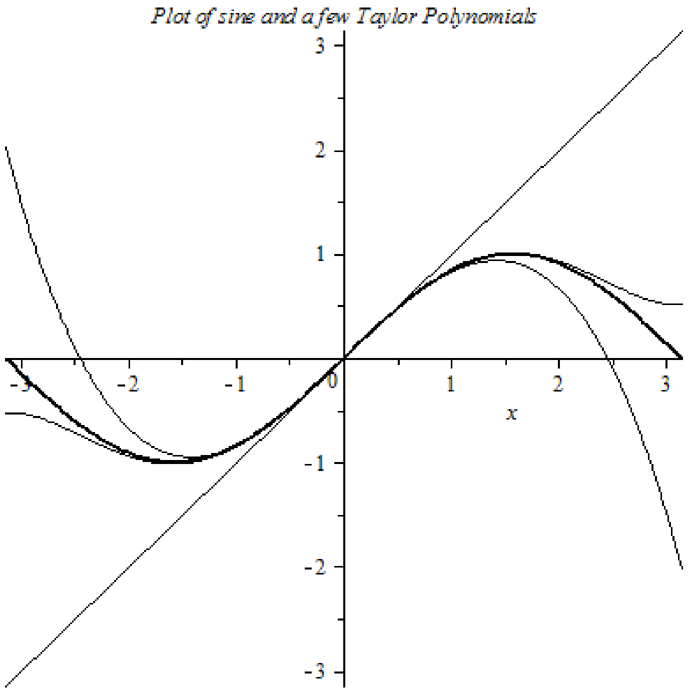
— —+…— 2.718281828459…

We know the error is

$E_{15}(1)$ —<—

Observations about Taylor Polynomial Approximations to f(x)

- Accurate near center – less accurate as move away from the center.
- Higher degree Taylor polynomials give more and more accurate approximations.



Application of Taylor Polynomials

- Approximate the integral:

)*dx*

- Use the Taylor Series

  ( ) — + — - — + …

- Substitute for *x*

  ( ) — + —— - —— + …

-)*dx* = [— — —— ——]

  = – —— —— ——

If we use just the first four terms we get approximately 0.3102681578.
Since this is an alternating series, the error is less than the next unused term, —— 1.45·10^-7