

Fixed Point Iteration

Chapter 2(Section 2.2)

Definition of fixed point

A **fixed point** for a function g is a value x such that $g(x)=x$.

Example: Let $g(x)=x^2-2$. Then 2 is a fixed point for g since $g(2)=2$.

Fixed Point Iteration: The iteration

$$p_{n+1} = g(p_n), n = 0, 1, 2, \dots$$

Theorem 2.1: If g is a continuous function and the iterative sequence $\{p_n\}, n=0, 1, 2, \dots$ converges to a finite number P , then P is a fixed point of $g(x)$.

Conditions for Existence of Fixed Point

Theorem 2.2: Assume g is a continuous function on a closed interval $[a, b]$.

If the image $g([a, b])$ is a subset of $[a, b]$, then g has a fixed point in $[a, b]$

If additionally g is differentiable on (a, b) and $|g'(x)| \leq K < 1$, then this fixed point is unique in $[a, b]$.

Sufficient Conditions for Convergence, Divergence of Fixed Point Iteration

Theorem 2.3 Fixed Point Theorem

Assume that g' is continuous on $[a,b]$ and $g([a,b])$ is a subset of $[a,b]$. Let p_0 be a point in $[a,b]$. Then

- i) If $|g'(x)| \leq K < 1$ for all x in $[a,b]$, then the iteration $p_{n+1} = g(p_n), n=0,1,2,\dots$ converges to a fixed point P .
- ii) If $|g'(x)| > 1$ for all x in $[a,b]$, then the iteration fails to converge.

Fixed Point Algorithm Pseudocode

- $P[1] = p_0$
- For $k = 2$ to max number of iterations
- $k = k+1$
- $P[k] = g(P[k-1])$
- if $|P[k] - P[k-1]|$ is small enough stop.
- End (For)