Fixed Point Iteration
Chapter 2(Section 2.2)

Definition of fixed point

A **fixed point** for a function g is a value *x* such that $g(x)=x$.

Example: Let $g(x)=x^2-2$. Then 2 is a fixed point for *g* since $g(2)=2$.

Fixed Point Iteration: The iteration
$p_{n+1}=g(p_n)$, $n = 0,1,2,\ldots$

Theorem 2.1: If *g* is a continuous function and the iterative sequence $\{p_n\}, n=0,1,2,\ldots$ converges to a finite number P, then P is a fixed point of $g(x)$.

Conditions for Existence of Fixed Point

Theorem 2.2: Assume *g* is a continuous function on a closed interval $[a,b]$.
If the image $g([a,b])$ is a subset of $[a,b]$, then *g* has a fixed point in $[a,b]$
If additionally *g* is differentiable on (a,b) and $|g'(x)| \le K< 1$, then this fixed point is unique in $[a,b]$.

Sufficien Conditions for Convergence, Divergence of Fixed Point Iteration

Theorem 2.3 Fixed Point Theorem
Assume that g' is continuous on [a,b] and g([a,b]) is a subset of [a,b]. Let p_0 be a point in [a,b]. Then

i) If $|g'(x)| \le K < 1$ for all x in [a,b], then the iteration pn+1=g(pn),n=0,1,2,… converges to a fixed point P.
ii) If $|g'(x)| > 1$ for all x in [a,b], then the iteration fails to converge.

Fixed Point Algorithm Pseudocode

- P[1] = p_0
- For k = 2 to max number of iterations
- k = k+1
- P[k] = g(P[k-1])
- if |P[k] –P[k-1] is small enough stop.
- End (For)