Numerical Analysis Lab Week Six Investigating Taylor Polynomials using MATLAB

1. The Taylor polynomial $P_n(x)$ of degree m centered at x_0 for the function $f(x)$ is defined by the formula

$$\sum_{k=0}^{n} f^{k}(x_0)\frac{(x-x_0)^k}{k!} = f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2!}+f'''(x_0)\frac{(x-x_0)^3}{3!}+\dots+f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

It is a good approximation to $f(x)$ near the center x_0.

2. To determine the Taylor polynomial requires the use of symbolic computations as opposed to numeric computations. MATLAB by itself does only numeric computations. To do symbolic computation, we must use the add-in product, the ***Symbolic Toolbox***, which actually uses the Maple kernel to do the symbolic calculations.

To use symbolic operations manipulating functions of x, we must create the symbolic variable x:

```
» syms x
```

3. Then we can do symbolic computations of expressions involving the symbol x such as

```
» diff(sin(x),x)
```

or just

```
» diff(sin(x))
```

4. To calculate the taylor polynomial of degree (order) 5 centered at 0 for sin(x) we enter:

```
» taylor(sin(x),6,0)
» pretty(ans)
```

The second command **pretty** just prints the output more nicely formatted.

5. In general, to calculate the Taylor polynomial of *f*(*x*) of order *m*-1, centered at *a*, we enter

taylor(f(x),a,m)

Thus to calculate the taylor polynomial for exp(x) (ex) centered at 1, of degree 6, we enter:

```
» taylor(exp(x),7,1)
» pretty(ans)
```

6. Create the Taylor polynomials for sin(x) centered at 0 of degree 1, 3, 5, and 7 with the following commands:

```
» t1 = taylor(sin(x),0,2)
» t2 = taylor(sin(x),0,4)
» t3 = taylor(sin(x),0,6)
» t4 = taylor(sin(x),0,8)
```

7. Using the **ezplot** function of the Symbolic Toolbox, we can plot both the sine function and several of its Taylor polynomials

```
» clf
» ezplot(sin(x))
» hold on;
» ezplot(t1)
» ezplot(t2)
» ezplot(t3)
» ezplot(t4)
```

Notice in each successive plot, how the *next higher degree Taylor polynomial becomes a better approximation* -- closer to sin(x) over a larger range about the center $x_0 = 0$.

8. Verify this by evaluating sin(x) and the Taylor polynomials at different points near 0. Fill in the table below, using commands such as

```
» sin(0.2)
» subs(t1,x,0.2)
```

x	sin(x)	t1 (Taylor degree 1)	t2 (Taylor degree 3)	t3 (Taylor degree 5)	t4 (Taylor degree 7)
0.2					
0.4					
3.0					

Note that none of the above are very good approximations at x = 3 because it is too far from the center. However a significantly higher degree Taylor Polynomial will still be good at x = 3. Calculate the Taylor polynomial of degree 19 centered at 0 and evaluate it at x = 3:

```
» t5 = taylor(sin(x),0,20)
» subs(t5,x,3)
```

Error Bounds for Taylor Polynomials

Exercise 1b Page 195

9. Suppose we want to determine an error bound for the 7th and 9th degree Taylor polynomials for *sin*(*x*) at *x* = 0 for the range -1 <= *x* <= 1 (exercise 1b, page 195).

Note: Factorial (!) is calculated with the factorial function
For example 6! is calculated with

```
» factorial(6)
```

Applying ***Theorem 4.1,*** page 189 , we need to calculate the maximum of the absolute value of the error term:

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{N+1}$$

-- first with N = 7, and then with N = 9.

We know the N+1'st derivative of sin(x) for N odd is +/-sin(x) and therefore is between +/-1 for all x. Hence we can bound the error in this case with:

$$|E_N| \leq \frac{|(x - x_0)^{N+1}|}{(N+1)!}$$

10. Thus we need to evaluate the right hand side of the inequality above for our given center $x_0 = 0$, and N = 7, and N = 9, and -1 <= *x* <= 1

Error bound for *n* = 7 ____________

Error bound for *n* = 9 ____________

Exercise 4, page 195

11. In addition to finding the Taylor polynomial (using MATLAB), also plot the Taylor polynomial, and the function on the same figure, using **ezplot (plot on the range 0 to 1 as below).**

a) Taylor Polynomial of degree N = 5 for $f(x) = 1/(1 + x)$ (as computed with **tp =taylor(..)** command):

tp =____

Plot as below:

```
>> ezplot(1/(1+x),[0,1])
>> hold on
>> ezplot(tp,[0,1])
```

b) Write down the Error term E_6 below: -- Note: Compute the 6th derivative for $f(x) = 1/(1 + x)$ using the symbolic **diff** command:

```
» diff(1/(1+x),6)
```

Error term ______________________

c) Give bound on the above error (as a function of x -- i.e. replace $f^{(N+1)}(c)$ with its maximum for c between 0 and x.

____

d) Evaluate your bound at $x = 0.3$ ___________________

e) Evaluate the function $f(x) = 1/(1 + x)$ and also its 5th degree Taylor polynomial (centered at 0) at $x = 0.3$, and compute the difference of these two at x = 0.3.

Difference ______________________________

f) Is the absolute value of their difference less than the error bound you gave? __________

Note: On TI92, TI89, and Voyage 200 calculators, the command to create a Taylor polynomial is:

```
taylor( expression, var, order, center)
```

e.g. for part e) `taylor(1/(1+x),x,5,0)`