Mth 439 Numerical Analysis Lab Issues in Numerical Matrix Computations: Ill Conditioned Matrices

A matrix is said to be ***ill-conditioned*** if relatively small changes in the entries of A cause relatively large changes in solutions to the system A**x** = **b**. If relatively small changes in the entries in A result in small changes in the solutions, it is said to be ***well-conditioned***. In order to predict their potential behavior, the condition number is used. If A is a nonsingular matrix, the ***condition number*** of A, denoted by c(A), is defined to be c(A) = ‖A‖ ‖A^{-1}‖, where ‖A‖ is a norm of the matrix A. Different norms may be used; one that *may* be used is

$$\|\mathbf{A}\| = \max\{|a_{i1}|+|a_{i2}|+\cdots+|a_{in}|\},\ i=1,\ldots,n$$

That is, add up the absolute value of each row; then take the maximum of the sums.

$$\mathrm{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -5 & 4 \\ 6 & 1 & -2 \end{pmatrix} \text{ with } \mathrm{A}^{-1} = \begin{pmatrix} 1/11 & 2/33 & 4/33 \\ 5/11 & -1/33 & -2/33 \\ 1/2 & 1/6 & -1/6 \end{pmatrix}$$

For example to compute c(A) (using the above norm) for the matrix A above:

‖A‖ = max{ 1 + 2 + 0, 3 + 5 + 4, 6 + 1 + 2 } = 12

‖A^{-1}‖ = max(1/11 + 2/33 + 4/33, 5/11 + 1/33 + 2/33 , 1/2 + 1/6 + 1/6) = 5/6

c(A) = (12)(5/6) = 10

In MATLAB, the ‖A‖ used in the previous example is computed by **norm(A,inf)** (called the infinity norm). Use **help norm** to see alternative norms that may be specified. MATLAB's **cond(A)** computes a condition number for A using the ***2-norm.***

1. Enter the following matrices, and compute their condition number using the condition number computed as **norm(A,inf)∗norm(inv(A),inf**)and also **cond(A).** Then compute the solutions to A**x** = **b** for the given **b**, and observe the results.

a)

$$\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2.005 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

norm(A,inf)∗norm(inv(A),inf) _____________

cond(A) _____________

A\b _____________

b)

$$\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2.01 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

norm(A,inf)∗norm(inv(A),inf) _____________

cond(A) _____________

A\b _____________

c) Compare the relative change in the entry $a_{2,2}$ in matrix A from part a) to part b) with the corresponding relative change in the solutions to A**x** = **b**.

2. Repeat the instructions in Exercise 1 for the following matrices:

a)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.005 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

norm(A,inf)∗norm(inv(A),inf) ____________

cond(A) ____________

A\b ____________

b)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.01 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

norm(A,inf)∗norm(inv(A),inf) ____________

cond(A) ____________

A\b ____________

c) Compare the relative change in the entry a_{22} in matrix A from part a) to part b) to the corresponding relative change in the solutions to A**x** = **b**.

3. A rule of thumb is for estimating accuracy of solutions of A**x** = **b** is:
If the condition number of A *is* $\approx 0.d \times 10^k$ *then the components of* **x** *can generally be expected to have k fewer significant digits of accuracy than the elements of* A.

If we assume that the change in the value of $a_{2,2}$ from part a to part b is rounding due to having limited significant digits of accuracy, the condition numbers should forecast the degree of error obtained in the answer for **x** from 1a) to 1b) and also 2a) to 2b). Write the condition numbers for A in 1a), 1b), 2a), and 2b) in the form $\approx 0.d \times 10^k$ (or compute **log10(cond(A))** in MATLAB), and explain the discrepancies in terms of this rule of thumb.