MTH 439 Numerical Analsysis Lab
Improvements to Gaussian Elimination:
LU (Triangular) Factorization with Permutations and the Gauss Seidel Iterative Method

m-files to download:
luquick.m gseid.m

Objectives

1. To demonstrate the method of LU factorization with Permutation of matrices and its usefulness.
2. To apply LU factorization to matrices.
3. To investigate the Gauss Seidel Iterative Method for solving linear systems.

LU Factorization

The ***LU factorization*** (also called triangular factorization or LU decomposition) of a matrix A is the factorization of the matrix as the product of a lower triangular matrix L (all zeros above the main diagonal), with1's on the main diagonal and an upper triangular matrix U (all zeros below the main diagonal).

An LU Factorization Method

If A can be reduced to an echelon form by elementary operations *without* row swapping, then we can find the factorization as A = LU,where the method for constructing L and U is as follows:

Reduce A to an echelon form U by a sequence of row operations (not using swapping). To construct the elements in L below the main diagonal, do the following: If the row operation required to obtain a 0 in the *i*th row under column *j* was "add -c times row *j* to row *i*", then set *l*ij = c. This is demonstrated with the m-file function **luquick.m**.

If a reduced echelon form requires row swapping, then a lower triangular matrix cannot be achieved. However, the method may be modified to produce a *permuted lower triangular matrix.* We consider this in the exercises that follow. This method is also used when partial pivoting is implemented to increase the accuracy of computations.

Exercise 1.

a) Use **[L U] = luquick(A)** (an accompanying demonstration M-file function) to compute the LU factorization for the following matrix. (Enter **format rat** first to set the format to rational).

$$A = \begin{pmatrix} 3 & 5 & 2 & 1 \\ 3 & -1 & 2 & 0 \\ 5 & 2 & 1 & 0 \\ 3 & 2 & 1 & -2 \end{pmatrix}$$

b) Verify the factorization by computing L*U (Should be equal to A, of course!)

Exercise 2.

Permutation matrices and MATLAB's* lu *command

A ***row permutation*** of a matrix is obtained by interchanging (permuting) one or more rows. This can be achieved by multiplying the matrix by a ***permutation matrix***. A permutation matrix is a matrix that is a row permutation of an identity matrix. (See text page 148). To permute (interchange) rows i and j of an mxn matrix, multiply the matrix on the left by the mxm permutation matrix constructed by permuting rows i and j of the identity matrix I_m.

a) Enter **help lu** for a description of MATLAB's LU factorization function. Note that it says the L in the command **[L U] = lu(A)** is the permutation of a lower triangular matrix -- the product of permutations matrices with a lower triangular matrix. Also, the command **[L U P] = lu(A)** returns a lower triangular L and upper triangular U and permutation matrix P such that P*A = L*U. These permutations may occur for purposes of reducing errors in computation (partial pivoting).

b) Using the matrix A in Exercise 1, enter the command

[L U] = lu(A)

Write down the matrices L and U that are returned:

c) Enter the command **[L2 U2 P] = lu(A)**. What is the relationship between L, L2, and P?

d) What is the relationship between U and U2?

Exercise 3:

If we use the factorization PA = LU, then we can solve the system AX = B with the following modified method for LU factorization:
1. Construct the matrices L,U, and P (as you did above in MATLAB)
2. Compute the column vector C = P*B
3. Solve LY = C for Y using forward substitution
4. Solve UX = Y for X using back substitution.

a) Perform these steps (using MATLAB) on the matrix A from Exercise 2 to solve the system AX = B,
where B is the column vector [23; 1; 15 ;3].

Since we don't want to actually do the forward and back substitution by hand, we will "cheat" and use **Y = L\C** for step 3 and **X = U\Y** for step 4.

C __________________ Y ____________________ X _____________________

b) Check your answer by asking MATLAB to solve directly with A\B.

A\B ___________________

Solve the system AX = B, where B = [3/2; -23/2; -17; -3] using the same decomposition above. (Of course you do not need to recompute L, U, and P)

C __________________ Y ____________________ X _____________________

Gauss Seidel

The **gseid.m** m file implements the Gauss-Seidel Iterative Method.

Exercise 4:

a) Enter the following commands that represent a linear system of equations AX=B

A = [2 8 -1; 5 -1 1; -1 1 4]; B = [11;10;3]

Write down system of equations represented:

b) Is it strictly dominant? Verify.

c) Find the “exact” solution using the command   **A\B**

d) Use Gauss-Seidel with 10 iterations to see what solution you get:

P = [0;0;0]
delta = 10^-9
max1 = 10
X=gseid(A,B,P,delta, max1)

d) Write down the values for the approximate solution **X** and then compute the relative approximate percent error for each.

f) Do a row swap on A and B to get a new equivalent linear system that is diagonally dominant.

What rows do you need to swap?

f) Next repeat Gauss-Seidel on this system with 10 iterations to see what solution you get.   Write down the values for X and then compute the relative approximate percent error for each.

Note: you can use gseid with 1,2, and 3 iterations to check parts of your homework set 6 to see if you are getting the correct answers.