6.1 Probability Density Functions

Def Let X be a random variable. If there exists a non-negative real-valued function f: R→[0, ∞) such that for any subset of real numbers A which can be constructed from intervals by a countable number of set operations,

$$P(X \in A) = \int_A f(x)dx.$$

Then f(x) is called the probability density function for X, and X is then called a ***continuous*** random variable. Generally, we just say f(x) is the density function for X.

Properties of Density Functions

$$F(t) = \int_{-\infty}^{t} f(x)dx.$$

$$F'(x) = f(t)$$

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

$$P(a < X < b) = P(a \le X < b)$$

$$= P(a < X \le b) = P(a \le X \le b) = \int_a^b f(t)dt.$$

Example 6.1

Experience has shown that while walking in a certain park, the time X, in minutes, between seeing 2 people smoking has a density function of the form

$$f(x) = \begin{cases} \lambda x e^{-x} & x > 0 \\ 0 & otherwise. \end{cases}$$

(a) Calculate the value of .
(b) Find the probability distribution function of X.
(c) What is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes? In at least 7 minutes?

Example 6.1 continued

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} \lambda x e^{-x}dx = \lambda \int_0^{\infty} x e^{-x}dx.$$

(b)

$$F(t) = \int_0^t f(x)dx = [-(x+1)e^{-x}]_0^t = 1-(t+1)e^{-t}.$$

(c) $P(2 < X < 5) = F(5) - F(2) = (1-6e^{-5}) - (1-3e^{-2}) = 0.37$

$P(X \ge 7) = 1 - F(7) = 0.007.$

Probability density functions

(b) $F(t) = 0$ if $t < 0$. For $t \ge 0$,

$$F(t) = \int_{-\infty}^{t} f(x)dx = [-(x+1)e^{-x}]_0^t = -(t+1)e^{-t} + 1.$$

(c) $P(2 < X < 5) = F(5) - F(2) = (1-6e^{-5}) - (1-3e^{-2}) \approx 0.37$

$P(X \ge 7) = 1 - F(7) \approx 0.007.$

Example 6.2

(a) Sketch the graph of the function

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}|x-3| & 1 \le x \le 5 \\ 0 & otherwise, \end{cases}$$

and show that it is the probability density function of a random variable X.
(b) Find F, the distribution function of X, and show that it is continuous.
(c) Sketch the graph of F.