Discrete Mathematics Chapter 5
Class Discussion Notes

[Section 5.1 The Basics of Counting]

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Section 5.1 The Basics of Counting

- **Example of Counting Problem**: How many possible passwords if they are allowed to be 6, 7, or 8 characters (letters or digits, not case sensitive).

- **Sum Rule**: The number of ways to perform choice of two tasks, where n1 is number of ways to perform task 1 and n2 is number of ways to perform task 2 , is equal to n1 + n2.
  - Example 1: Can choose either a math faculty or student for committee.
  - Obvious extension to arbitrary number of tasks.

- **Product Rule**: A task (or decision) consists of a sequence of two tasks. First perform task 1 and then perform task 2. The total number of different ways to perform the task is n1*n2 where n1 is the number of ways to perform task 1 and n2 is the number of ways to perform task 2.
  - Obvious extension to arbitrary number of tasks.

  - Example 2: Number of chairs in auditorium with letter and a digit. 26*10.
  - Example 3: 32*24 ports.
  - Example 4: Bit strings of length seven. Choose first bit, second bit, etc. 2^7.
  - Another example:
    - Choosing Pres, Vice-Pres., Secretary from a group of 10 people.
      - -- where can hold more than once office
      - -- where must be distinct.
  - Example 10: Number of Subsets of a Finite Set S of size n. (Set of all subsets of S is called the power set of S -- asking for cardinality of the Power Set).

    - Regard as a sequence of decisions about each element in the set -- either put it in the set or do not put it in the set. Like the bit string problem.
  - Password Example
    - Number of passwords where they must be alphanumeric and 6 characters long (not case sensitive).
    - Number of passwords that do not contain any digits (alphabetic only)
    - Number of passwords that contain 6 alphanumeric characters with at least one digit and are
      - 7 characters long
      - 8 characters long
- **Counting using the Inclusion-Exclusion Principle:**
  - Applies when counting the number of ways to do ***either of two tasks*** where there is ***possible overlap*** between the two tasks.
  - **Example 15:** How many bit strings of length eight either start with a 1 bit or end with two zero bits (or both)
    - First Task: Construct a bit string of length 8 that start with a 1 bit
    - Second Task: Construct a bit string of length 8 that end with two 00 bits.
    - Overlap: Constructing a bit string that both starts with a 1 bit and ends with two 00 bits.
    - **Total number of way to complete task** = Number of ways to first task + Number of ways to do the second task - Number of overlaps between the two tasks.
    - **Answer to example 15**: 2^7 + 2^6 - 2^5

Inclusion-Exclusion Principle phrased in terms of sets:
Number of elements in A1 union A2 = Number of elements in A1 + Number of elements in A2 - Number of elements in A1 intersect A2.

Tree Diagrams:

  - Used to model problems regarded as a sequence of tasks where each task represents making a choice. We want to count the number of possible different sequences of choices.
  - Extensively used in game theory.
  - Number of ways to perform sequence of tasks is the number of distinct paths from the root to a leaf (i.e. the number of leaf nodes) in the tree.
  - **Examples**:
    - **Figure 2**: Bit Strings of Length Four without Consecutive 1s.
    - **Figure 3:** Best of Five Series.

Section 5.2: The Pigeonhole Principle

- **Theorem 1: (Basic Pigeonhole Principle)** If k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

  - **Example:** 100 pairs of of socks (4 different colors). At most how many do you have to pull out of a drawer until you get a pair of socks to match.
  - See **examples 1, 2, and 3.**

- **Theorem 2: The Generalized Pigeonhole Principle**

  If N objects are placed into k boxes, then there is at least one box containing at least ceiling(N/k) boxes.

  - **Examples 4, 5, 6, page 348**
  - **Exercise Number 32 (page 354):** Each computer can be connected to either 1, 2, 3, 4, or 5 others. Only five possible choices for each of 6 computers.

Section 5.3 Permutations and Combinations

Factorial Notation: n! = 1*2*3*...*n. 0! is defined to be 1.

Permutations

- **Permutation of a set of objects:** An ordered arrangement of the objects.
- **r-permutation of a set of objects :** An ordered arrangement of size r of the objects.
  - **Example 2**: Set S = {1, 2, 3}
    - Permutation example: (3, 1, 2)
    - 2-permutation: (3, 1)
- **Number of r-permutations of a set of n distinct elements** is

  P(n,r) = n(n-1)(n-2) ...(n-r+1)
  Alternative formula: n!/(n-r)!

- **See Example 4:** Selecting 1st, 2nd, 3rd prize winners from 100

- Example 7: Permutations of ABCDEFGH containing ABC -- treat "ABC" as a single block of letters.

Combinations

- An **r-combination of elements** of a set is unordered collection of r elements from the set. (Since it is unordered an r-combination can be treated as a subset of size r).
- **See Example 8, page 357:** Number of ways to make committee of size 3 from 4 students.
- **Number of r-combinations** of a set with n distinct objects is denoted by C(n,r) and is calculated as

$$\mathbf{C(n,r) = n!/(r!(n-r)!)}$$

- *Note*: This is also called the ***binomial coefficient*** (since occurs in the expansion of $(a + b)^n$