Chapter 2 The Fundamentals: Algorithms, Integers, and Matrices

[Section 2.1 Sets]

[Section 2.2 Set Operations]

[Section 2.3 Functions]

[Section 2.4 Sequences and Summations]

Section 2.1 Sets

- **Set** is a collection of objects (duplicates are not allowed)
  Example: Set of students in Discrete Mathematics.
- Objects in the set are called **members** or **elements of the set.**
- **Set notation**: Members are listed in { } separated by commas. -- {1, 2, 3, 4}

  The order in which they are listed is not important. {1, 2, 3, 4} is the same as {4, 3, 1, 2}

  See examples 1, 2, 3

- Common sets: natural numbers, integers, positive integers, rational numbers, real numbers.

- Two sets are **equal** if they have the same elements.
- **Set builder notation**: { x | P(x) } where P(x) is a propositional function.

  Example: {x **|** x is odd OR x is less than 10 }

- **Universal set:** Denote by **U.** Set of all possible elements under consideration .
- A is a **subset** of B if every element of A is an element of B
  {1, 2, 3} is a subset of {1, 2, 3, 4, 5} (in fact a ***proper subset***)
- **Venn diagrams** are often used to indicate relationships between sets -- see figures 1 and 2, pp 114-115
- **Null set** -- denoted by Greek letter phi. Also called the empty set and denoted by { } -- consists of no elements.

- Sets can have *other sets* as members -- { ϕ, {a}, {b}, {a, b}}
- A is a **proper subset** of B if it is a subset of B but not equal to B.
- **Cardinality of a set**: Number of distinct elements of a set.
  See examples 9, 10, 11
- A set is **finite** if it has n distinct elements for some finite integer n. Otherwise it is **infinite**. The set of integers is an infinite set.
- **Power Set:** Set of all subsets of a set. Note that the empty set is always considered to be a subset of every set. Example: What is the power set of {a, b, c}?
- **Cartesian Product** of two sets A and B : The set of all ordered pairs (a, b) where a is an element of A and b is an element of B.
  - Useful for database applications
  - See examples 15, 16, 17

Section 2.2 Set Operations

- **Union:** Examples 1 and 2, page 121
- **Intersection:** Examples 3 and 4, page 122
- Two sets are **disjoint** if their intersection is the empty set.
- **Difference A - B:** Examples 6 and 7, page 123
- **Complement** of a set A. Examples 8, 9, page 123-124
- **Venn Diagrams --** figures 1, 2, 3, 4, 5 pp 122-123, 127

Set Identities

- See table 1 page 89 for common set identities.
- One way of showing a set identity is with membership tables (similar to truth tables).
  - See example 13 page 91
  - List the basic sets and possible combinations of memberships for any element (1 is in the set, 0 is not in the set).

Note: Not covering generalized unions and intersections.

Computer representation of finite sets.

- If U is a finite set of reasonable size n, can use a bit string of length n to represent any subset of U.
- See examples 19, page 129
- Set operations: Complement is formed by taking bitwise NOT; Union is bitwise OR, Intersection is bitwise AND. (see example 20, p 130)

Fuzzy Sets: (See exercises 61-62 pages 132-133)

- Used in artificial intelligence. Each element in U has a degree of membership in fuzzy set S.

Section 2.3 Functions

- **Def: Function**. A function *f*: A -> B is a mapping (or assignment) for each element *x* of A to a unique element in B (denoted by *f(x)*).
    - A is called the **domain** of *f*
    - B is called the **codomain** of *f*
    - *f*(a) is called the **image of a**. If f(a) = b then a is called the **preimage** of b.
    - The **range** of *f* is the set of all images of elements of the domain A.
- See figure 1 for an example of a function which maps each student in a class to a letter grade.
- **Def: One-to-one**. A function *f* is one-to-one (or injective) iff f(x) = f(y) implies that *x* = y for all *x* and *y* in the domain of *f.*
    - See figure 3 page 137 for an example of a one-to-one function.
    - Figure 1 (grading example) is not one-to-one
    - Example 9: $f(x) = x^2$ is not one-to-one
    - Example 10: $f(x) = x+1$ is one-to-one
- **Def:** Function is **strictly increasing** if $f(x) < f(y)$ whenever $x < y$.
- **Def:** Function is **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$.
- Note: Strictly increasing (or decreasing) functions are always one-to-one.
- **Def: Onto**. A function *f* from A to B is onto (or surjective) iff for every element *b* in B there is an element in A with $f(a) = b$.
    - Figure 4, page 138 is an example of an onto function
    - Example 12: $f(x) = x^2$, where the domain and codomain are the set of integers, is **not** onto.
    - Example 13: $f(x) = x + 1$ is onto.
- A function that is both one-to-one and onto is called a **bijection** or **one-to-one correspondence**.
- See figure 5, page 139, for examples of different types of correspondences.
- Two real-valued functions with the same domain can be added and multiplied:

  Let f_1 and f_2 *be* two functions from domain A to codomain R (the set of real numbers). The sum and product functions are defined by:
    - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
    - $(f_1 * f_2)(x) = f_1(x) * f_2(x)$

Inverses of Functions and Compositions of Functions

- **Def: One-to-one correspondence.** A function that is both one-to-one and onto is called a one-to-one correspondence.
- **Def: Inverse function.** If f is a one-to-one correspondence from A to B, then the inverse function, f^{-1}, is the function that assigns to an element *b* in B the unique element *a* in A such that $f(a) = b$.

- One-to-one correspondences are called **invertible** and functions which are not one-to-one correspondences are not invertible.
- See example 16 and 17, page 140 for inverses of invertible functions.
- Note that $f(x) = x^2$ is not invertible.

- **Def: Composition** of functions. Let g be a function from A to B and f a function B to the set C. The composition of the functions is a function from A to C and with value f(g(a)) for a in A.
  - See figure 7, p 141 for a diagram of the idea of composition.
  - See examples 20 and 21, page 141 for examples of compositions. Note that composition is NOT commutative

The Graphs of Functions

- **Def: Graph of a function f** is { (a, f(a)) **|** a is in the domain of f }. Normally we represent by a picture of the graph using the Cartesian coordinate system.
- See examples 22 and 23 (figures 8 and 9), page 142
- Special functions:
  - **floor function**: Floor (x), where x is a real number, is the greatest integer less than or equal to x.
  - **ceiling function**: Ceiling(x) is the smallest integer than is greater than or equal to x.
  - Floor (3.6) = 3. Ceiling(3.6) = 4. See further examples in example 24 and graphs in figure 10, page 143.
- **Exponential Functions** (See Appendix 1) $f(n) = b^n$ is the exponential function to the base b. If n is an integer, defined to be b^n is defined to be b multiplied by itself n times.
  - Properties:
    - $b^{x+y} = b^x b^y$
    - $(b^x)^y = b^{xy}$
- **Logarithmic Functions** (See Appendix 1). The **logarithmic function base b** is the inverse of the exponential function. In other words $\log_b(x)$ is the power to which b must be raised to get x.
  - Examples: $\log_2(16) = 4$; $\log_2(32) = 5$; $\log_{10}(1000) = 3$.
  - Important Properties:
    - $\log_b(xy) = \log_b(x) + \log_b(y)$
    - $\log_b(x^y) = y \log_b(x)$
    - $\log_a(x) = \log_b(x)/\log_b(a)$ (Change of base formula)
- See figures 1 and 2 in Appendix 1 for graphs of exponential and log functions.

Section 2.4 Sequences and Summations

Definition 1: A *sequence* is a function from a subset of integers (usually either the nonnegative integers or the positive integers) to a set S. Use subscript notation: a_n is the value of the sequence (function) at integer n.

Example 1: $a_n = 1/n$, n= 1, 2, 3, ...

Definition 2: A geometric progression (or geometric sequence) is a sequence of the form a, ar, ar^2, ..., ar^n where a and r are constants.
It is an exponential function whose domain is restricted to the nonnegative integers.

See example 2, page 150 for geometric sequences.

Definition 3: An arithmetic progression (or arithmetic sequence) is a sequence of the form a, a + d, a + 2d, ..., a + nd, where a and d are constants. This is a linear function y = a + dx where x is restricted to the nonnegative integers.

See example 3: -1 + 4n and 7 - 3n

Special Integer Sequences:
Recognizing a formula or general rule for constructing terms of a sequence.
Things to look for:
Are there runs of the same value?
2, 2, 2, 3, 3, 3, 4, 4, 4,
Are there terms obtained from previous terms by adding the same amount on an amount that depends on n?
5, 8, 11, 14, ...
Are there terms obtained by previous terms by multiplying by a certain amount?
3, 6, 12, 24, 48, ...

Are terms obtained by combining previous terms in a certain way?
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Are there cycles among the terms?
1, 2, 3, 1, 2, 3, 1, 2, 3

Example 5:

1, 1/2, 1/4, ...	$a_n = 1/2^n$, n= 0, 1, 2, ...
1, 3, 5, 7, 9	$a_n = 2n +1$, n= 0, 1, 2, ...
1, -1, 1, -1	$a_n = (-1)^n$, n= 0, 1, 2, ...

Example 7:

5, 11, 17, 23, 29, 35, ... $a_n = 5 + 6n$, n= 0, 1, 2, ...

Example 8:

1, 7, 25, 79, 241, 727, 2185, 19681, 59047

Differences: 6, 18, 54, 162, ??? -- tripling each time?

$a_n = 3^n - 2$

Summations:

Summations notation is shorthand way of writing sum:

Giving lower limit, upper limit, and terms in terms of index of summation.

Example 9: (page 154) sum of first 100 terms of 1/n, n = 1, 2,3 ...

Example 10, 11, 12: Evaluating sums. Expand summation notation, then add up terms.

Theorem 1: Formula for geometric series. (Sum of terms of geometric progressions).

$a_0 + a_1 r + a_2 r^2 + \ldots\ldots a_n r^n = (ar^{n+1} - a)/ (r - 1)$

Example 13: (page 156) Double summations. First expand inner summation (simplify if possible), then expand outer sum.

Summations where index varies *over a set*: See **Example 14,** page 156

Table 2, page 157 gives some useful summation formulas (don't have to memorize)

(Skip section on cardinality)