

Chapter 1: Foundations of Discrete Mathematics: logic, sets, functions

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Section 1.1 Logic

- **Discrete Mathematics:** Study of discrete objects (distinct, unconnected, countable) (as opposed to points on a line -- of which there is an uncountably infinite number)
- **Logic** . Provides foundation for mathematical reasoning
- **Proposition:** Statement that declares a fact that is either TRUE or FALSE.
 - Examples of propositions vs. non-propositions (Examples 1 and 2 page 2)
 - Conventional letters for variables that represent propositions are p, q, r, s, \dots
 - T, F are used for truth values TRUE, FALSE
- **Compound propositions:** Formed from existing propositions using logical operators.
- **Truth tables** are used to show truth values of compound propositions given truth values of propositions from which they are constructed.
- $\neg p$ **Negation of p** (not p – denoted by) -- "It is not the case that p "

Example 3 page 3

Truth Table - page 3 Table 1

False when p is true, True when p is false

- $p \wedge q$ **Conjunction of p and q .** " p and q ".

Example 4 page 4

Truth Table - page 5 Table 2

True when both p and q are true; false otherwise

- $p \vee q$ **Disjunction of p and q .** " p or q " (inclusive or).

Example 5 page 5

Truth Table - page 5 Table 3

False when both p and q are false, true otherwise

- $p \oplus q$ **Exclusive or.** " p or q , but not both"

True when p and q exactly one of p and q is true, and false otherwise.

Truth Table - page 6 Table 4

- $p \rightarrow q$ **Implication.** " p implies q "; "if p then q "
 - False when p is true and q is false, true otherwise.
 - p is the **hypotheses** (antecedent or premise), q is the **conclusion** (consequence)
 - Truth Table - page 6 Table 5
 - Alternative ways of expressing implication: (page 6)

- if p then q
 - p implies q
 - if p, q
 - p only if q
 - p is sufficient for q
 - a sufficient condition for q is p
 - q if p
 - q whenever p
 - q is necessary for p
- Helpful examples:
 - Politician: "If I am elected then I will lower taxes"
 - Professor: "If you get 100% on the final, then you will get an A."
- Note the following:
 - "If this year is 1900, then 2+3 = 5" has truth value of TRUE (since the conclusion is always TRUE).
 - "If this year is 1900, then 2+3 = 6" has truth value of TRUE (since the hypothesis).
 - "If Elvis is alive, then I'm the Queen of England" has the truth value of TRUE
 - "If Elvis is dead, then I'm the Queen of England" has the truth value of FALSE
- Don't confuse with the **programming language construct** if p then S (where S is a statement to be executed -- as in Example 8 , page 8:

```
x := 0
if 2 + 2 == 4 then x := x + 1.
```

- **Converse** of implication $p \rightarrow q$: $q \rightarrow p$
- **Contrapositive** of implication $p \rightarrow q$: $\neg q \rightarrow \neg p$
- **Inverse** of implication $p \rightarrow q$: $\neg p \rightarrow \neg q$

Implication and its contrapositive have the same truth value (i.e. are logically equivalent).

Converse and inverse have the same truth value (i.e. are logically equivalent).

- See Example 9. "**The home team wins whenever it is raining.**" To find the converse, contrapositive, and inverse, write in form $p \rightarrow q$, and identify p and q.
- **Biconditional: $p \leftrightarrow q$** True when both p and q have same truth values, false otherwise. That is, it is true precisely when both the implication and its converse are both true. Read as "p if and only if q" or "p iff q"
 - Truth Table: page 9 Table 6 \
 - Example 10, page 9: You can take the flight if and only if you buy a ticket
- Precedence of logical operators -- In compound propositions, which operators should be evaluated first?
 - Parentheses force the order of evaluation.
 - When parentheses are not used: order of precedence (from highest to lowest) is not, and, or, implication, biconditional.

Truth Tables for Compound Propositions

- Construct truth tables by forming rows corresponding to possible combinations of truth values for each of the propositional variables. Build truth values of subexpressions following rules of precedence.
- See Example 11, Table 7, page 10

Translating English sentences into logical expressions:

- Example 12, page 11 "You can access the Internet from campus only if you are a computer science major or if you are not a freshman." $a \rightarrow (c \vee \neg f)$
- Example 13, page 11 " You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old ".

q: You can ride the roller coaster
r: You are under 4 feet tall.
s: " You are older than 16 years old."

$$(r \wedge \neg s) \rightarrow \neg q$$

System Specifications

- Example 14: "The automated reply cannot be sent when the file system is full"

p: "The automated reply can be sent."
q: "The file system is full."
 $q \rightarrow \neg p$

- Example 15: Are system specifications **consistent** -- can we assign truth values to propositions in the expressions that make all of the compound propositions true.
 - The diagnostic message is stored in the buffer or it is retransmitted. $p \vee q$
 - The diagnostic message is not stored in the buffer. $\neg p$
 - If the diagnostic message is stored in the buffer then it is retransmitted. $p \rightarrow q$
 -

p: "The diagnostic message is stored in the buffer."
q: "The diagnostic message is retransmitted."
Must let p be F (so $\neg p$ is T); Hence q must be T (so $p \vee q$ is T)
 $p \rightarrow q$ will be T (since $F \rightarrow T$ is T)
Hence system is consistent.

Logic and Bit Operations

- bit: binary digit -- 0 or 1.
- bitwise operations correspond to the logical operators where 0 is false, and 1 is true.
- **OR, AND, XOR, NOT**
- Bit string -- string of bits. length is number of bits in string. Apply bitwise operations to corresponding bits.
- Example 21, page 15-16.

Section 1.2 Propositional Equivalences

- **Tautology:** Compound Proposition that is always true no matter what the truth value of the propositions in it.
 - Example **p OR NOT p**.
- **Contradiction:** Compound Proposition that is always false.
 - Example **p AND NOT p**.
- **Contingency:** Compound Proposition that is neither a tautology nor a contradiction
- **Logically Equivalent:** Two propositions p and q are logically equivalent ($p \equiv q$) if the biconditional $p \leftrightarrow q$ is a tautology (is always true). (Equivalently, they have the same truth values.)
- **Method for showing two compound propositions are logically equivalent:** Just set up the truth tables for each and show they have the same truth values.
- Examples to look at:
 - Tables 1,2,3, 4, 5 page 22-23
- Table 6, 7, 8 page 24-25 -- some important logical equivalences (don't have to memorize)
- Example 5 page 25 shows how to use DeMorgan's laws to negate expressions.
- Example 6 and 7, page 26 shows how compound propositions can be shown to be logically equivalent by replacing logical equivalences.
- NAND and NOR
 - NAND: True when either p or q or both are FALSE, false when both are true.
 - NOR true when both p and q are false, false otherwise.

Section 1.3 Predicates and Quantifiers

- **Propositional Function:** A proposition which has a variable in it (something that can have different values assigned to it).
 - Denote by $P(x)$
 - Consist of subject (variable) and **predicate** (what you're saying about the subject variable). Example: $P(x): x > 3$
 - Can have more than one variable: Example: $Q(x,y): x = y + 3$.
 - Interested in evaluating the truth values for different values of the variables. $P(2)$; $Q(7,4)$

Quantifiers

- **Universe of discourse** for a proposition specifies all of the possible values the variable can take on.
- Two important definitions:
 - **Universal quantification of P(x)**

$$\forall x P(x)$$

For all values of x in the universe of discourse, P(x) .

- For all x, P(x)
- For every x, P(x)
- **Existential quantification of P(x)** There exists an x (at least one) in the universe of discourse, P(x)

$$\exists x P(x)$$

- There exists an x such that P(x)
 - For some x P(x)
-
- **Examples:** Let the universe of discourse be the set of all integers.
 - For every x, $x^2 \geq 0$ (TRUE)
 - For every x, $x^3 \geq 0$ (FALSE)
 - There exists x such that $x^3 \geq 0$ (TRUE)
 - Also **examples 1 -- 13.**
 - Table 1, page 34 -- when quantifiers are TRUE or FALSE
 - For every x, P(x). True if P(x) is true for every x; False if there exists at least one x for which P(x) is false.
 - There exists x, P(x). True if there is at least one x for which P(x) is true. False if P(x) is false for every x.

Negating Quantified Expressions:

Look at Table 2, page 41 for negating quantifiers. Know the equivalent negations.

Example 20:

Negation of " There is an honest politician"

Negation of " All Americans eat cheeseburgers".

Translating English expressions into logical expressions

- Must identify universe of discourse, subject variable x, and predicate.
 - Example 23, page 42: "Every student in this class has studied calculus."
 - Example 24, page 42: "Some student in this class has visited Mexico."
"Every student in this class has visited either Canada or Mexico."
 - Not one unique way -- depends partly on what is chosen as universe of discourse.

Section 1.5 Rules of Inference

- **Rules of Inference:** Forms of argument that can be used to draw conclusions from other assertions.
- **Fallacy:** Incorrect form of an argument. (A false argument).
- The following tautologies are basic rules of inference used in proofs.

Rules of Inference

$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$	Conjunction
$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$	Addition
$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$	Simplification
$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$	Modus ponens
$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	Modus tollens
$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	Hypothetical syllogism
$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	Disjunctive syllogism
$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$	Resolution

- Examples 1, 2, 3, 4, 5 pp 57-58
- Other examples: (What rules of inference are used?)

If I study hard, I will get an A
 I study hard.
 Therefore I will get an A.

If you read all of your Core books,
 you are be very dedicated.
 If you are very dedicated,
 you impress your professor.
 You read all of your Core books.
 Therefore you impress your professor.

My eyes are not red.
 If I drink, my eyes are red.
 Therefore I did not drink.

n is an even number.
 If n is an even number, then n^2 is an even number.
 Therefore n^2 is an even number.

x is greater than 10
 Therefore either x is greater than 10 or less than -10.

x is less than 20 and greater than 12.
 Therefore x is less than 20.

Either you take the final exam or you fail the course.
 You did not take the final exam.
 Therefore you fail the course.

Valid argument: An argument is valid if whenever all of the hypotheses are true, the conclusion will also be true. Note a valid argument can lead to an incorrect conclusion if one or more of the hypotheses are false. A valid argument uses known results and rules of inferences.

See examples 6 and 7 for valid formal arguments using rules of inference.

- **Common fallacies:** Arguments based what look like a rule of inference but are not correct.
 - **Fallacy of affirming the conclusion.**

$$\frac{p \rightarrow q \quad q}{p}$$

- **Example:**

If you have been drinking your eyes are red.
Your eyes are red.
Therefore you have been drinking.

- **Fallacy of denying the hypothesis.**

$$\begin{array}{r} p \rightarrow q \\ \neg p \\ \hline \neg q \end{array}$$

- **Example:**

If you have been drinking your eyes are red.
You have not been drinking.
Therefore your eyes are not red.

Example 6

- Suppose we have the following hypotheses:
 - (1) "It is not sunny and it is cold."
 - (2) "We will swim only if it is sunny."
 - (3) "If we do not swim, then we will canoe."
 - (4) "If we canoe, then we will be home early."
- Given these hypotheses, prove the conclusion "We will be home early" using rules of inference.

Abbreviations:

sunny = "It is sunny"; *cold* = "It is cold";
swim = "We will swim"; *canoe* = "We will canoe"; *early* = "We will be home early".

Premises can be written as:

- (1) $\neg \textit{sunny} \wedge \textit{cold}$
- (2) $\textit{swim} \rightarrow \textit{sunny}$
- (3) $\neg \textit{swim} \rightarrow \textit{canoe}$
- (4) $\textit{canoe} \rightarrow \textit{early}$

Conclusion can be written as:

early

Formal Argument:

<u>Step</u>	<u>Reason</u>
1. $\neg \textit{sunny} \wedge \textit{cold}$	Hypothesis #1.
2. $\neg \textit{sunny}$	Simplification of step 1.
3. $\textit{swim} \rightarrow \textit{sunny}$	Hypothesis #2.
4. $\neg \textit{swim}$	Modus tollens with step 2,3.
5. $\neg \textit{swim} \rightarrow \textit{canoe}$	Hypothesis #3.
6. <i>canoe</i>	Modus ponens with step 4,5.
7. $\textit{canoe} \rightarrow \textit{early}$	Hypothesis #4.
8. <i>early</i>	Modus ponens with step 6,7.

Table 2 page 70

Universal Instantiation and Generalization

Existential instantiation and Generalization

Universal Modus Ponens

$\forall x$, if $P(x)$ then $Q(x)$
 $P(a)$ for a particular a

$\therefore Q(a)$.

Universal Modus Tollens

$\forall x$, if $P(x)$ then $Q(x)$
 $\neg Q(a)$ for a particular a

$\therefore \neg P(a)$.

Section 1.6 Introduction to Proofs

Introduction

Basic Terms:

- **Theorem:** Statement that can be shown to be true.
 - **Lemma:** a simple theorem used to help prove a more significant theorem.
 - **Corollary:** a result that is easily derived from a theorem.
- **Proof:** Sequence of statements that form an argument to show a theorem or other result is true.
- **Conjecture:** A statement whose truth value is unknown. (When proved, becomes a theorem).

Axioms (or postulates): Statements that are accepted as true and used as foundations in proofs.

Methods of Proving Theorems

- **Direct Proof:** Showing $p \rightarrow q$ is true by showing that if p is true, then q must also be true.

Definitions: The integer n is **even** if there exists an integer k such that $n = 2k$.
The integer n is **odd** if there exists an integer k such that $n = 2k + 1$.

Example 1: If n is odd then n^2 is odd.

- **Indirect Proof:** Showing $p \rightarrow q$ by showing $\neg q \rightarrow \neg p$ is true. (That is, prove the contrapositive directly).

Example 3: If $3n + 2$ is odd where n is an integer, then n is odd.

- **Definitions:** A real number r is **rational** if there exists integers p and q with q not equal to 0 such that $r = p/q$. A real number that is not rational is called **irrational**.

Example 7: Direct proof that the sum of two rational numbers is rational

Example 8: Indirect proof that for n an integer, if n^2 is odd, then n is odd.

- **Proof by Contradiction:** Showing p is true by showing $\neg p \rightarrow q$ is true where q is known to be false (q is called the contradiction). (This works, since this implies that $\neg p$ must be false and so p is true).

Example 10: Prove that square root of 2 is irrational by proof by contradiction. The proof shows that assuming sqrt of 2 is rational leads to a contradiction.

- **Proofs of Equivalence -- biconditionals (If and only if proofs)**

To prove $p \leftrightarrow q$ must show both $p \rightarrow q$ and $q \rightarrow p$. (Hence requires two proofs).

Example 12: Prove n is odd if and only if n^2 is odd. (Both parts already given in Example 1 and Example 8, respectively)