MTH 122 Discrete Mathematics Proof by Mathematical Induction

Mathematical induction is a method of proof. It is used to prove statements of the form:

$\forall$ integers n $\geq$1, P(n)

where P(n) is a propositional function. Note this means that P(n) is a statement that evaluates to true or false for a given value of n.

(Acutally the integer **1** above can be replaced with any integer -- it is the smallest integer for which the result holds. The most frequent other starting integer is 0.)

Examples of statements that can be proved by Mathematical Induction:

1. $\forall$ integers $n \geq 1$**,** $1 + 3 + 5 + \ldots + (2n-1) = n^2$
2. $\forall$ integers $n \geq 0$, $n < 2^n$

It is based on the Principal of Mathematical Induction (expressed as a rule of inference):

$$\begin{array}{l} P(1) \\ \forall \text{ integer } k \quad P(k) \rightarrow P(k+1) \\ \hline \therefore \forall \text{ integers } n \geq 1, \quad P(n) \end{array}$$

An inductive proof always requires the following two steps:

1. **The Basis Step:** Prove P(1) is true (Usually very easy)
2. **The Inductive Step:** Prove P(k) $\rightarrow$ P(k+1) is always true.

Do the Inductive Step by assuming P(k) is true and showing that P(k + 1) is true.
(Note: P(k) is called the *inductive hypothesis*).

Note: *If you are trying to prove* **$\forall$ integers n >= 0, P(n),** *then the Basis Step is to prove P(0) instead of P(1).*

Hints for figuring out the inductive step: 1) Write down what P(k) is. 2). Write down what P(k + 1) is. Try to figure out how they differ -- i.e. how to derive P(k + 1) from P(k).

1. If P(k) is an equality involving a sum of k terms, the inductive step is usually proven by adding the k+1'st term in the sum to each side of the equality P(k). Then show that gives you P(k+1) by simplifying the right hand side.

2. If P(k) is an inequality often you may need to do one or more of the following to show P(k+1)

a) Add a number (or inequality in the same direction) to each side
b) Multiply each side by a positive number, or an inequality in the same direction.)
c) Divide each side by a positive number, (or an inequality in the same direction.)

Example

(This is example 1 in your text)

Prove that the sum of the first n odd positive integers is n^2.

That is, prove $\forall\, n \geq 1,\ 1 + 3 + 5 + \ldots + (2n-1) = n^2$

Equivalently, in summation notation:

$$\text{Prove } \forall\, n \geq 1,\ \sum_{i=1}^{n}(2i-1) = n^2$$

Proof:

Let P(n) denote the proposition that $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

1. Basis Step:

P(1) is true since $1 = 1^2$

2. Inductive Step:

Assume P(k) is true, that is

(*) $1 + 3 + 5 + \ldots + (2k-1) = k^2$

Adding $2(k+1) -1$ to each side of the equation (*) give

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2k-1) + [2(k+1) -1] &= k^2 + [2(k+1) -1] \\ &= k^2 + 2k + 1 && \text{(simplify)} \\ &= (k + 1)^2 && \text{(factor)} \end{aligned}$$

so P(k+1) is true.

By the principle of mathematical induction, P(n) is true for all integers $n \geq 1$.
Q.E.D.

MTH 122 Discrete Mathematics
Proof by Mathematical Induction Worksheet

Name __________________________

Problem 1: Prove using mathematical induction:
$\forall\, n \geq 0,\ 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

Brainstorming: (This is just thinking – not part of actual proof)

P(0): ____

P(k): ____

P(k+1): ____

We can get P(k+1) from P(k) by ____

Proof: (All of what follows is what should be written in the proof)

Let P(n) denote the proposition _______________________________________

1. Basis Step: P(0) is true, since

____

2. Inductive Step:

Problem 2: Example 2 in text
Prove using mathematical induction that $n < 2^n$ for all positive integers n.

Brainstorming:
P(1): ________________

P(k): ________________

P(k+1): ________________

We can get P(k+1) from P(k) by adding the inequality ____________________ to P(k).

Proof:
Let P(n) denote the proposition that $n < 2^n$.

1. **Basis Step**:

P(1) is true, since ___________________________

2. **Inductive Step**:

Assume P(k) is true, that is
(*) ___________________________

Then adding 1 to each side of (*) gives

____

so P(k+1) is true.

By the principle of mathematical induction, P(n) is true for all $n \geq 1$.
Q.E.D.

Problem 3: Example 5: Geometric Progression

Prove: $a + ar + ar^2 + \cdots + ar^n = \dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$, for all $n \geq 0$

Brainstorming:

P(0): ________________________________

P(k): ________________________________

P(k+1): ______________________________

Proof:

Let P(n) denote the proposition that __________________________

1. Basis Step: P(0) is true, since

____

2. Inductive Step: