

# MTH 122

## Discrete Mathematics

### Semester 112

1

## Why Study Discrete Math?

- For the computer scientist:
  - ⌘ Digital computers are based on discrete bits (binary digits). Discrete Mathematics is used to model both the structure of a computer hardware and the design and analysis of algorithms that run on a computer.
- For the mathematician:
  - ⌘ Discrete Mathematics provides a foundation of logic, set theory, logical argument, number theory, linear algebra and matrices, and probability (The non-calculus foundation)

4

## Instructor – Karen Donnelly

- ⌘ Office: Core 257
- ⌘ My web site: [www.saintjoe.edu/~karend](http://www.saintjoe.edu/~karend)
- ⌘ Office Phone: 866-6297
- ⌘ E-Mail: [karend@saintjoe.edu](mailto:karend@saintjoe.edu)
- ⌘ Course Web site: [www.saintjoe.edu/~karend/m122/](http://www.saintjoe.edu/~karend/m122/)

2

## Topics

- Logic
- Set Theory
- Functions
- Sequences
- Number Theory
- Mathematical Reasoning
- Counting
- Probability Theory
- Relations and Equivalence Relations
- Graphs and Trees

5

## Grading Policy

### Grade Distribution:

Assignments, Quizzes:	25%
Four Tests:	45%
Final Exam:	25%
Attendance, Participation:	05%

### Grading Scale:

93%-100%	A	90%-92%	A-		
87%-89%	B+	83%-86%	B	80%-82%	B-
77%-79%	C+	73%-76%	C	70%-72%	C-
67%-69%	D+	60%-66%	D		
59% or Below	F				

3

## Chapter 1: The Foundations

### Logic and Methods of Proof

6

## Propositions

- Logic is based on propositions.
- A proposition is a statement (*declarative sentence*) that is either true or false .
- The truth value of a proposition is either true (T) or false (F).

7

## Not a Proposition Example

$$"x > 1"$$

Is this a statement?            yes

Is this a proposition?        no

The truth value is not known (assuming that the value of x has not been specified).

Actually "propositional function" addressed in section 1.3

10

## Example of Proposition

⊗ "The Bears will not play in Superbowl 2012."

Is this a statement?            yes

Is this a proposition?        yes

What is the truth value  
of the proposition?        true  
(unfortunately)

8

## Compound Proposition Example

⊗ "Today is Monday and it is 80 degrees outside."

Is this a statement?            yes

Is this a proposition?        yes

What is the truth value  
of the proposition?        false

11

## Example of Proposition

$$"1 + 1 = 4"$$

Is this a statement?            yes

Is this a proposition?        yes

What is the truth value  
of the proposition?        false

9

Example: A sentence that is not a statement

⊗ "Read you Discrete Math book tonight."

Is this a statement?            no

Is this a proposition?        no

Not a proposition, since only statements can be propositions.

12

### Proposition Examples

⊗ "If Elvis is alive,  
⊗ Then I am the queen of England."

Is this a statement?            yes

Is this a proposition?            yes

What is the truth value  
of the proposition?            true

(More on this Later)

### Negation (NOT) $\neg$

⊗ Unary Operator

⊗ Truth Table

p	$\neg p$
true	false
false	true

16

### Compound Propositions

One or more propositions together with logical operators can be combined to form a compound proposition.

Traditionally use letters p, q, r, s as symbols for propositions.

14

### Conjunction (AND) $\wedge$

⊗ Binary Operator

⊗ Truth Table

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

17

### Logical Operators (Connectives)

- Negation             $\neg$     NOT
- Conjunction         $\wedge$     AND
- Disjunction         $\vee$     OR
- Exclusive or         $\oplus$     XOR
- Implication         $\rightarrow$     if - then
- Biconditional       $\leftrightarrow$     if and only if

⊗ Truth tables can be used to show how these operators can combine propositions to compound propositions.

15

### Disjunction (OR) $\vee$

⊗ Binary Operator

⊗ Truth Table

p	q	$p \vee q$
true	true	true
true	false	true
false	true	true
false	false	false

18

### Exclusive Or (XOR) $\oplus$

⊗ Binary Operator  
⊗ Truth Table

p	q	$p \oplus q$
true	true	false
true	false	true
false	true	true
false	false	false

19

### Biconditional (if and only if) $\leftrightarrow$

⊗ Binary Operator  
⊗ Truth Table

p	q	$p \leftrightarrow q$
true	true	true
true	false	false
false	true	false
false	false	true

22

### Implication (if – then) $\rightarrow$

⊗ Binary Operator  
⊗ Truth Table

p	q	$p \rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

20

### Statements and Operations

⊗ Statements and operators can be combined in any way to form new statements.

p	q	$p \wedge q$	$\neg (p \wedge q)$
true	true	true	false
true	false	false	true
false	true	false	true
false	false	false	true

23

### English Phrases Equivalent to $p \rightarrow q$

<ul style="list-style-type: none"> <li>• "p implies q"</li> <li>• "if p, then q"</li> <li>• "if p, q"</li> <li>• "when p, q"</li> <li>• "whenever p, q"</li> <li>• "q if p"</li> <li>• "q when p"</li> <li>• "q whenever p"</li> </ul>	<ul style="list-style-type: none"> <li>• "p only if q"</li> <li>• "p is sufficient for q"</li> <li>• "q is necessary for p"</li> <li>• "q follows from p"</li> <li>• "q is implied by p"</li> </ul>
--	---

### Converse, Inverse, Contrapositive

For an implication  $p \rightarrow q$ :

- ⊗ *converse* :  $q \rightarrow p$ .
- ⊗ *inverse* :  $\neg p \rightarrow \neg q$ .
- ⊗ *contrapositive*:  $\neg q \rightarrow \neg p$ .

Note: an implication and its contrapositive are logically equivalent.

### Contradiction

⊗: A compound proposition that is always false. Example  $p \wedge (\neg p)$

$p$	$\neg p$	$p \wedge (\neg p)$
true	false	false
false	true	false

25

### Logical Equivalence Example

$p$	$q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
true	true	true	false	false	false	false
true	false	false	true	false	true	true
false	true	false	true	true	false	true
false	false	false	true	true	true	true

⊗ The statements  $\neg(p \wedge q)$  and  $(\neg p) \vee (\neg q)$  are logically equivalent since they have the same truth value.

### Tautology

⊗: A compound proposition that is always true. Example  $p \vee (\neg p)$

$p$	$\neg p$	$p \vee (\neg p)$
true	false	true
false	true	true

26

### Boolean Operators

Boolean operations can be extended to operate on bit strings:

	01 1011 0110
	<u>11 0001 1101</u>
Bit-wise OR	11 1011 1111
Bit-wise AND	01 0001 0100
Bit-wise XOR	10 1010 1011

### Logical Equivalences

Two propositions are logically equivalent if they have the same truth values.

Symbol for logical equivalence is  $\Leftrightarrow$

Important Examples: DeMorgan's Laws

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$

$$\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q).$$

27

### Predicate Logic

- ⊗ In the sentence "Sam is sleeping":
  - ⊗ The word "Sam" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - ⊗ The phrase "is sleeping" denotes the *predicate* - a property that is true of the subject.
- ⊗ In *predicate logic*, a *predicate* is modeled as a *function*  $P(x)$  whose value at  $x$  is a proposition.
  - ⊗  $P(x)$  = "x is sleeping" (where  $x$  is any object).

## Quantifiers

- ⊗ **Universe of discourse:** *domain for all values of a variable  $x$  in a propositional function  $P(x)$ .* (We shall abbreviate as u.d.)
- ⊗ **Universal quantifier:**  $\forall$  – FOR ALL
  - ⊗  $\forall x P(x)$  means for all  $x$  in the u.d.,  $P$  holds.
  - ⊗ True when  $P(x)$  is true for every  $x$ .
  - ⊗ False when there is at least one  $x$  for which  $P(x)$  is false.
- ⊗ **Existential quantifier:**  $\exists$  EXISTS
  - ⊗  $\exists x P(x)$  means there exists an  $x$  in the u.d. such that  $P(x)$  is true.
  - ⊗ True when  $P(x)$  is true for at least one  $x$  in u.d.
  - ⊗ False when  $P(x)$  is false for every  $x$  in u.d.

## Universal Modus Tollens

$$\frac{\forall x, \text{ if } P(x) \text{ then } Q(x) \quad \neg Q(a) \text{ for a particular } a}{\therefore \neg P(a)}$$

- ⊗ Example:
  - ⊗ Everyone who likes cheese also likes pizza.
  - ⊗ Jane doesn't like pizza.
  - ⊗ Therefore Jane doesn't like cheese.

## Negating Quantifiers

- ⊗  $\neg \forall x P(x)$  logically equivalent to  $\exists x \neg P(x)$
- ⊗  $\neg \exists x P(x)$  logically equivalent to  $\forall x \neg P(x)$

## Universal Instantiation

$$\frac{\forall x, P(x)}{\therefore P(c)}$$

(Here  $c$  is a particular object in the universe of discourse).

- ⊗ Example:
  - ⊗  $U$  = professional quarterbacks.
  - ⊗ Every professional quarterback makes big money.
  - ⊗ Therefore Peyton Manning makes big money.

## Universal Modus Ponens

$$\frac{\forall x, \text{ if } P(x) \text{ then } Q(x) \quad P(a) \text{ for a particular } a}{\therefore Q(a)}$$

- ⊗ Example:
  - ⊗ Everyone who likes cheese also likes pizza.
  - ⊗ Jane likes cheese.
  - ⊗ Therefore Jane likes pizza.

## Existential Generalization

$$\frac{P(c) \text{ for some } c}{\therefore \exists x P(x)}$$

- ⊗ Example:
  - ⊗ Assume universe of discourse  $U$  is the set of professional quarterbacks.
  - ⊗ Peyton Manning makes big money.
  - ⊗ Therefore there is a professional quarterback who makes big money.