Linear Growth vs. Exponential Growth and Decay

Math as a Human Pursuit Lab

Put your name here

Important -- Do this before the rest of the commands: Place cursor anyway on line below and press Enter to execute the command:

```
> restart; with(plots) :
```

Introduction

You have just won the grand prize in a lottery. The lottery commission tells you that you may choose from one of two cash prizes: Either

Option 1: $1,000 per day for 30 days

Option 2: $0.01 (one penny) the first day, $0.02 the next day, $0.04 the next day....for 30 days. Each day's pay is double the pay from the previous day.

Which would you choose? Answer by entering the expressions in Maple below to represent amount of money for each at the end of 30 days.

Option 1: Income would be:

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>
```

Option 2: Income in pennies would be:

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>
```

Therefore you choose:

Answer:

Linear Growth

Linear growth occurs over time has a **constant** amount each unit of time. Therefore when we graph the growth as a functuon of time, we have a straight line graph. The first option for the lottery represents linear growth.
The slope of linear graph represents this change per unit of time (in the above example the slope is $1000). With linear growth, the formula for quantity Q at time t is given by:

$$Q = Q_0 + m\,t$$

where Q_0 is the initial quantity, and m constant amount of increase per unit of t.
We can plot this linear growth with command below:

> $plot(1000\,t, t = 0\,..30)$

Example of Linear Growth

Example of Linear Growth

Saint Joseph's college tuition currently is $25,080 per year. If it increases $500 per year, what will it be in three years? What will it be in 24 years? The calculations in Maple are shown below:

> $24350 + 500 \cdot 3$

> $24350 + 500 \cdot 24$

> $plot(24350 + 500\,t, t = 0\,..24)$

Exponential Growth Function

Exponential growth occurs when a quantity grows by the same percentage (proportion or rate) each time period.
Exponential decay occurs when a decreases by the same percentage each time period.

Exponential Growth/Decay Formula

Initial quantity Q_o

Time t

r = fractional growth (or decay if negative) rate per unit of time (must be same units used for a t).
Then the quantity Q at time t is given by:

$$Q = Q_0\,(1 + r)^t$$

In the Lottery Example Option 2 in the introduction, the value for ***r*** is 1 (100%), Q_0 is equal to .01 dollars (1 cent), and the time t represents the numbers of days after day one. Hence we can calculate the number of pennies received on day 30 as

> $0.01\;2^{29}$

A plot of this exponential growth is given by:

> $plot(0.01\;2^t, t = 0\,..29)$

On what day would you first receive a payday of over 1 million dollars?. (Figure by trial and error -- replace WHAT with a number and press enter.)

Answer:

> $.01 \cdot 2^{\mathrm{WHAT}}$

Exponential Population Growth Example

Below we define the function *Q* which represents population growth (in millions) of the United States population since the year 2000, assuming an exponential growth with a fractional growth rate of r = .007 . After plotting, vary the growth rates (by change the .007 to higher values - then re-execute each statement (try for example .009, .010, .020 and observe the changes -- replotting each time).

> $Q := 281\ (1 + 0.007)^t$

> $subs(t = 100, Q)$

> $plot(Q, t = 0\,..100)$

The following is an animation of the dramatic effect of varying the growth rates from r = .002 to r = .040 (*don't worry about how this command works - it is somewhat complicated*). Put the cursor on the anywhere on line below and press enter to display the plot. Then click on the graph to select it, then press the "play" arrow. (Make sure that you have executed the command at the beginning of the lab to load the plot library).

> $A := display\left(seq\left(plot\left(281\ \left(1 + \frac{r}{1000}\right)^t, t = 0\,..60\right), r = 2\,..40\right), insequence = true\right); \%$

>

Human population growth isan important example of exponential population growth. The boom in population growth today began with the Industrial Revolution (See Section 8C of your text for more information).

http://opr.princeton.edu//popclock/

The following link takes us to the United States Census Bureau's data about world population growth.

http://www.census.gov/ipc/www/idb/

We can see from the charts that the growth rate *r* is in fact not a constant value and has in fact been declining.

Using the chart on World Population Growth Rates, estimate the population growth rate (as a decimal) for the year 2010. Then use that in the population formula to estimate the population in 2050.

> $Q := \mathrm{WHAT}\ (1 + .\mathrm{WHAT})^{\mathrm{WHAT}}$

What is unrealistic about the predicted population that we calculated? Compare this to the predicted population in 2050 at this Web site.

Answer:

Exponential Doubling Time

Doubling time: The time required for an exponentially growing quantity to double. In the option 2 lottery example, the doubling time was t = 1 day. If you know the doubling time T_{double} for an exponentially growing quantity, you can compute the quantity at time T by the formula:

$$Q = Q_0 \, 2^{\frac{t}{T_{double}}}$$

where Q is the quantitaty at time t (new value) and Q0 is the initial quantity (initial value).
Assume exponential growth for the world population with a doubling time T = 50 years. Given the population of the world this year 2010, what would the population be in 10 years? What would the population be in 100 years?

Answers:

>

>

Exercises using Exponential Growth Function

Exercises:
1. When I went to the local movie theater (what is now The Ritz) as a child, I took 25 cents -- 15 cents to get in the movie, 5 cents for popcorn, 5 cents for coke or candy bar. This was approximately 50 years ago.
Assuming an annual inflation rate of 2% how much would that be equivalent to today?
Answer:

>

2. My first car as college student was a 1968 (?) Camaro which cost (I think) $2200 or so.
Assuming an annual inflation rate of 2%, how much would that be equivalent to today.
Answer:

>

3. Suppose that you borrow on your credit card $3000. Your credit card charges 1.5% **per month** interest, compounded monthly. Assuming that you make no payments and no new charges -- and no penalties are assessed, what do you owe at the end of 1 year, 2 years, 5 years?
Answer:

Bonus:
How much would you owe if you also are fined $25 per month for not making the minimum payments?)
Answer:

>

Exponential Decay

If **r is negative** the we have **exponential decay** (declining values for Q). Example: China's Goal for Declining Population. China has a goal of reducing its population to 700 million by 2050. In the year 2000 (treating this as our starting year), the population was 1.2 billion. Will a rate of decline of -0.5% allow them to reach their goal? When will they reach their goal of 1.2 billion with this rate of decline? (Replace each WHAT with the appropriate numbers)

```
> Q := 1.2·(1 − WHAT)^t
> subs(t = WHAT, Q)
> plot(Q, t = 0 ..120)
```

Exponential Decay and Half Life

Half life: In exponential decay, the time it takes for the quantity to decrease to half of its original amount.

If we know the half-life in an exponential decay model, we can calculate the population at any time using the following formula, where T_{half} is the half life, t is the elapsed time, and Q_0 is the initial quantity.

$$Q = Q_o \left(\frac{1}{2}\right)^{\frac{t}{T_{half}}}$$

The **proportion** remaining after time *t* would then be computed as:

$$\text{Proportion remaining } \frac{Q}{Q_0} = \left(\frac{1}{2}\right)^{\frac{t}{T_{half}}}$$

Suppose for example that 200 pounds of Plutonium, Pu-329, which has a half life of 24,000 years, is deposited at a nuclear waste site. How much will still be present after 1000 years? What percentage is remaining?

Answer:

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>
>
```

Radiometric Dating Formula. To find out how old an item is, we analyze the amount of radioactive material remaining. (See the example 7 page 544 on the Allende Meteorite). We can find the age (elapsed time t) using our formula above and solving for t.

$$t = \frac{T_{half} \log\left(\frac{Q}{Q_0}\right)}{\log\left(\frac{1}{2}\right)}$$

Compute the age of the rocks in Exercise 43 and 44:

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