The Four-Colour Problem

a presentation by Mike Koscielny

Historical Perspectives of Mathematics

What is a Graph?

- A graph is an abstract mathematical structure.
- It is used to model relations between given objects from a collection.
- The interconnected objects—called vertices—are commonly diagrammed as dots, or points, with lines, or curves, connecting them as edges.
- Graphs, as typically encountered in algebraic mathematics, are ordered pairs of integers plotted on a set of axes.

Graph Theory?

- The first recorded "graph theory problem" is the the paper written by Leonhard Euler on the Seven Bridges of Königsberg (1736).
- Differential Calculus and Topology helped develop the study of graph theory.
- The Four Colour Problem is arguably the most famous problem of graph theory.
- The problem itself is credited being first posed by Francis Guthrie in 1852, then being first written down is in a letter from Augustus De Morgan addressed to William Rowan Hamilton in the same year.

Colour Mapping?

- The colouring of maps is something of more interest to mathematicians than actual mapmakers.
- Mathematicians have found that three colours are adequate for simpler maps, but a fourth colour is required for certain types of maps. (EX: some maps in which a region is surrounded by an odd number of other regions that touch each other in a circle.)
- A simple proof by contradiction can prove that five colours can be used to colour regions without sharing a border.

The Four-Colour Problem? Theorem?

- Can any map drawn in a single plane have its regions coloured with four colours in such a way that any two regions having a common border will always have different colours?
- Or, given a map drawn, you can use just four colours to fill in every single region and no region of a colour will share a border with a region also filled with the same colour.

Prove It?

- Proving the four-colour theorem is difficult. Multiple attempts and false proofs have been offered since the problems inception.
- It was first proved in the year 1976 by two men, named Kenneth Appel and Wolfgang Haken from the University of Illinois.
- Historically, it was **the *FIRST* major theorem to be proved using a computer.**

Proof With A Computer?

- Appel and Haken used an approach which started by showing there is a specific and particular set (containing 1936 maps) of which each map could not be part of a smallest-sized counterexample to the four-colour theorem.
- Part of their logic stated that if, in fact, the four-colour conjecture were false, there would be at least one map with the smallest possible number of regions that requires five colours. From there, the proof they used showed that such a minimal counterexample cannot exist, through the use of two concepts.

The Logic In The Proof?

- One, the idea of an "unavoidable set," which contains regions such that every map must have at least one region from this collection.
- Two, the idea that reducible configurations can be made from certain maps. A reducible configuration is a map which can be reduced to a smaller map.
- Therefore, the smaller map has the condition that if it can be coloured with four colours, then the original map can also. This implies if the original map can not be coloured with four colours, the smaller map cannot either, therefore the original map is not minimal.

The Logic In The Proof? (cont.)

- Using the rules and procedures based on properties of reducible configurations, Appel and Haken found an unavoidable set of reducible configurations, thus proving a minimal counterexample could not exist.
- Their proof reduced the infinitude of possible maps to 1,936 reducible configurations, which had to be checked one by one by computer and took over a thousand hours.
- For the sake of certainty, the reducibility part of the work was independently double checked with different programs and computers.

A Simpler Way?

- Since the proving of the theorem, efficient algorithms have been found for four-colouring maps requiring only $O(n^2)$ time, where n is the number of vertices.
- In 1996, Thomas (et al) created a quadratic time proof algorithm to improve on the quartic algorithm used previously.
- This new proof is similar to the original, but more efficient because it reduced the complexity and required checking 633 reducible configurations. Both the unavoidability and reducibility parts must still be executed by computer because they are too impractical to check by hand.

Works Referenced?

- Appel, Kenneth, Wolfgang Haken, and John Koch. "Every Planar Map is Four Colorable." *Bulletin of the American Mathematical Society 82.5 (1976): n. pag. Web. 26 Apr 2010. <http://projecteuclid.org/DPubS/Repository/1.0/Disseminate?view=body&id=pdf_1&handle=euclid.bams/1183538218>.*
- Thomas, Robin. "An Update on the Four-Color Theorem." *Notices of the American Mathematics Society 45.7 (1998): 848-59. Web. 26 Apr 2010. <http://www.ams.org/notices/199807/thomas.pdf>.*
- WIlson, Robin. *Four colors suffice: How the Map Problem was Solved. Princeton, NJ: Princeton University Press, 2004. Print.*

QUESTIONS